Doubled geometry and string theory

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Outline

★ Introduction

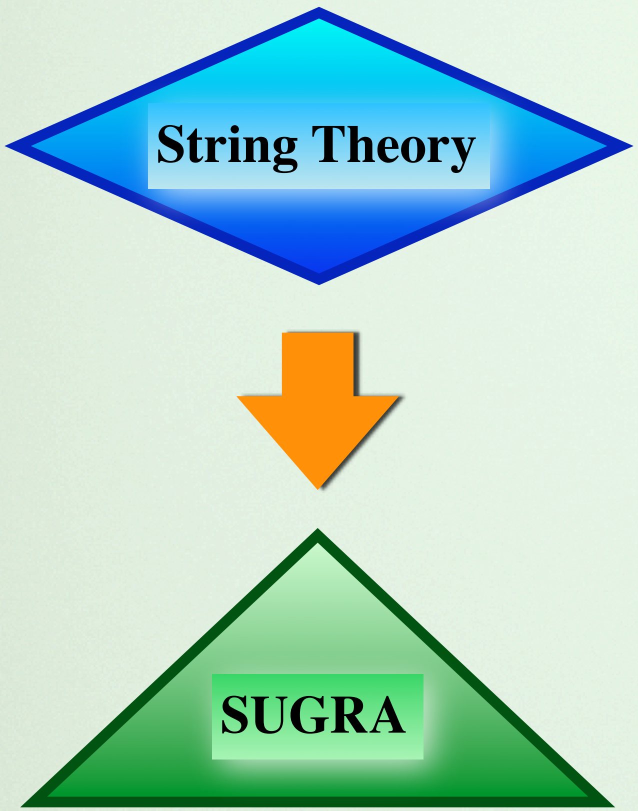
★ T-folds

★ Doubled formalism

★ D-branes

★ Comparison with generalised geometry

Introduction



String theory needs 10 dim

→ Compactify on 6-dim space to get 4-dim theory

→ Massless scalars parameterising internal space

But no massless scalars in nature

→ make scalars massive: **Moduli stabilization**

Moduli stabilization

Turn on background fluxes on internal space

$\longrightarrow$ potential that stabilises moduli & generates mass

But:

T-duality leads to non-geometric spaces

- OK, because some of them are consistent string backgrounds

Ex: T-folds, asymmetric orbifolds

Hull ‘04: Doubled formalism

Geometric description of T-folds

↓

Transition functions in T-duality group

Generalise → **Doubled geometry** [Hull ‘06]

Geometric description of more general non-geometric spaces

Doubled geometry

Includes all T-duals of a given nonlinear sigma model, in a single description.

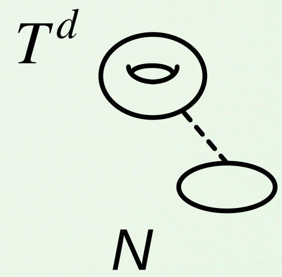
Involves constructing a “doubled space” & a “doubled sigma model”.

From this obtain the (mutually dual) physical models.

$\longrightarrow$ Better understanding of T-duality?

T-folds

Take torus fibration M:

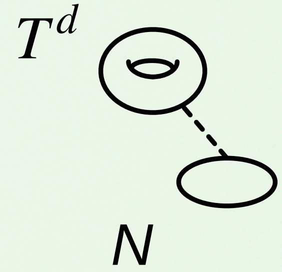


General background tensor: $E_{ij} = G_{ij} + B_{ij}$

(legs along fibres, independent of fibre coords)

T-folds

Take torus fibration M:



General background tensor: $E_{ij} = G_{ij} + B_{ij}$

(legs along fibres, independent of fibre coords)

- For **geometric** transition functions: $h \in GL(d;\mathbb{Z})$

(large diffeomorphisms on T^d)

In overlap $U \cap U'$ of patches on base: $E' = h E h^T$

- For **non**-geometric transition functions:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d;\mathbb{Z})$$ (T-duality symmetry on T^d fibration)

$\longrightarrow$ g preserves $L = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$

In overlap of patches on base: $E' = (a\,E + b)\,(c\,E + d)^{-1}$

$\downarrow$

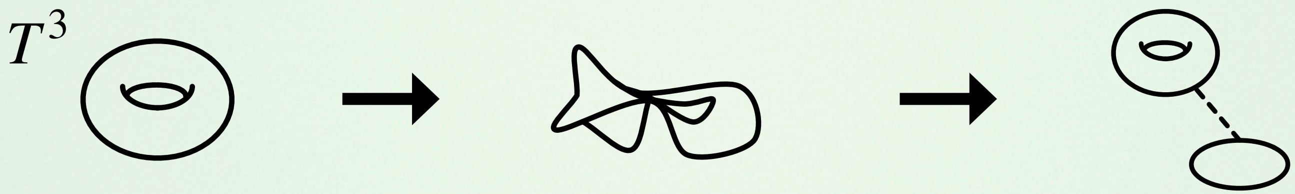
mixes G & B

T-duality monodromy
around non-contractible
loop in the base $\longrightarrow$ T-fold = mondrofold

<u>Ex:</u> $N = S^1$, $Y \sim Y + 2\pi$

$\Rightarrow$ $E(Y + 2\pi) = (a\,E(Y) + b)\,(c\,E(Y) + d)^{-1}$

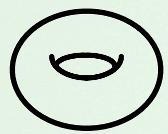
T-folds arise as T-duals of flux compactifications:



H-flux	f-flux	Q-flux
	(twisted torus)	(T-fold)
H = dB	B = 0	
geometric	geometric	non-geometric

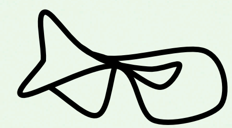
Locally a T^2 fibration over S^1
but globally twisted by $O(2,2;\mathbb{Z})$

T^3



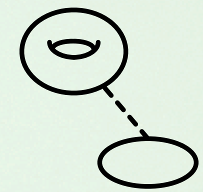
T-duality
along z

$\longrightarrow$



T-duality
along y

$\longrightarrow$



$$B = \frac{m}{3}\,(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$$

$$H = dB = m\,dx \wedge dy \wedge dz$$

$$B = 0$$

$$ds^2 = dx^2 + dy^2 + \left(dz + \frac{m}{3}\,x dy - \frac{m}{3}\,y dx\right)^2$$

$$h = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

T^2 fibration over S^1 with geometric monodromy $h \in GL(2;\mathbb{Z})$

Global twist $g \in O(2,2;\mathbb{Z})$

- Example of T-fold: The mirror of a Calabi-Yau with NS-NS flux on its T^3 fibres

- Analogously can construct S-folds (S-duality), U-folds (U-duality) and mirror-folds (mirror symmetry).

Doubled formalism

String theory compactified on T^d:

$\longrightarrow$ moduli space $O(d,d)/O(d) \times O(d)$

$p_{L,R} = p \pm w$ take values in Narain lattice

momentum

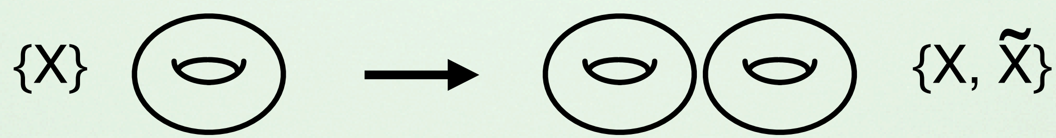
winding

Conjugate coords: $X_{L,R} = X \pm \tilde{X}$

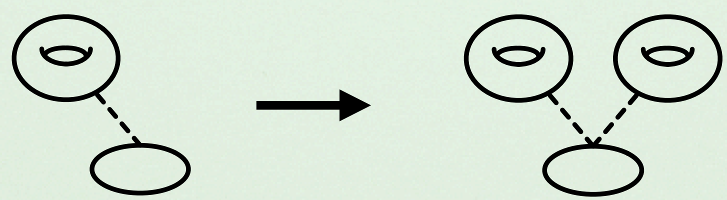
dual coord

The T-dual of T^d is also a T^d

→ Natural to formulate as string theory on target space T^{2d} with $O(d,d;\mathbb{Z})$ T-duality acting on its coordinates X, $\tilde{X}$



T-fold:



Degrees of freedom are doubled: $\{X\} \to \{X, \tilde{X}\} \equiv \{\mathbb{X}\}$

→ Impose constraint:

worldsheet
Hodge dual

$$dX_L = {}^*dX_L \qquad dX_R = -\,{}^*dX_R$$

→ $\begin{cases} dX = {}^*d\tilde{X} \\ d\tilde{X} = {}^*dX \end{cases}$ **Self-duality constraint**

→ Usual T-duality transformations

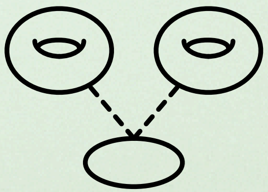
Defines natural O(d,d) invariant metric L on T^{2d}:

$\{X\}$ and $\{\tilde{X}\}$ spaces are *maximally null* w.r.t. $L = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$

T-duality group $O(d,d;\mathbb{Z}) \subset GL(2d)$ = group of large diffeomorphisms of T^{2d}

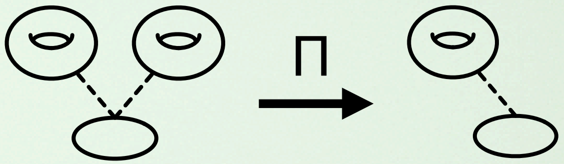
→ The T-duality monodromy of T^d over N has become a geometric transition function for T^{2d} over N.

→ The new space is a T^{2d} fibration over N.



Polarisation

To recover physical target space (the T-fold), introduce projectors:



$$\Pi = \frac{1}{2}(\mathbb{1} + R) \qquad \tilde{\Pi} = \frac{1}{2}(\mathbb{1} - R)$$

projects to physical space

projects to dual space

R = product structure on T^{2d} fibres: $R^2 = \mathbb{1}$

⟶ Splits tangent space into ±1 eigenspaces of R

Necessary properties of R:

- ★ $R \in GL(2d; \mathbb{Z})$

- ★ Metric L is pseudo-hermitian w.r.t. R:

$$L_{IK}\, R^K{}_J + L_{JK}\, R^K{}_I = 0$$

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Then:

- Π & $\tilde{\Pi}$ are null w.r.t. L: $\Pi^T L \Pi = \tilde{\Pi}^T L \tilde{\Pi} = 0$

  $\longrightarrow$ Π & $\tilde{\Pi}$ define *maximally null* subspaces

- R is preserved by $GL(d; \mathbb{Z}) \subset O(d,d;\mathbb{Z})$

Suppose $\exists$ product structure R_α on patch $U_\alpha \subset N$.

- R_α is integral $\Rightarrow$ constant over U_α but can have $R_\alpha \neq R_\beta$

on $U_\beta \subset N$

- R_α defines splitting of fibres over U_α : $T^{2d} \to T^d \oplus \tilde{T}^d$

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If: $R_\alpha = g_{\alpha\beta}^{-1} R_\beta \, g_{\alpha\beta}$, $g_{\alpha\beta} \in GL(d; \mathbb{Z})$ (geometric case)

then +1 eigenspaces fit together to form a T^d bundle over N

and $R_\alpha = R_\beta \longrightarrow$ globally defined polarisation

But for T-fold: $R_\alpha = g_{\alpha\beta}^{-1} R_\beta \, g_{\alpha\beta}$, $g_{\alpha\beta} \in O(d,d;\mathbb{Z})$

$\Rightarrow$ $R_\alpha \neq R_\beta$ $\longrightarrow$ $\exists$ only local polarisation

$\Rightarrow$ T^d 's do not patch together to a manifold

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$\Rightarrow$ T^d 's do not patch together to a manifold

Action of $g_{\alpha\beta} \in O(d,d;\mathbb{Z})$:

Physical space defined by R_β $\longrightarrow$ Physical space defined by $R_\alpha = g_{\alpha\beta}^{-1} R_\beta g_{\alpha\beta}$

$h \in GL(d;\mathbb{Z})$ preserving R_β $\longrightarrow$ Conjugate $GL(d;\mathbb{Z}) \subset O(d,d;\mathbb{Z})$ with elements $g\, h\, g^{-1}$

Maximally null T^d $\longrightarrow$ Another max null T^d

Sigma model

Consider local patch $U \times T^{2d}$:

- Coordinates: $\{Y^m, \mathbb{X}^I\}$, $I = 1,\ldots,2d$
- Covariant momenta on T^{2d} : $\mathcal{P}^I = \mathcal{P}^I{}_\alpha \, d\sigma^\alpha$

worldsheet coords

- Bianchi identity: $d\mathcal{P}^I = 0$

$\longrightarrow$ Locally $\mathcal{P}^I{}_\alpha = \partial_\alpha \mathbb{X}^I$

- Introduce metric on T^{2d} : $\mathcal{M}_{IJ}(Y)$
  - independent of $\mathbb{X}^I$
  - positive definite

Sigma model on $U \times T^{2d}$:

$$\boxed{\mathcal{L} = \frac{1}{2}\, \mathcal{M}_{IJ}\, \mathcal{P}^{I} \wedge {}^{*}\mathcal{P}^{J} + \mathcal{L}\,(Y)}$$

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$$\boxed{\mathcal{L} = \frac{1}{2}\,\mathcal{M}_{IJ}\,\mathcal{P}^{I} \wedge {}^{*}\mathcal{P}^{J} + \mathcal{L}\,(Y)}$$

? How to choose the self-duality constraint?

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Sigma model on $U \times T^{2d}$:

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? How to choose the self-duality constraint?

! Together with Bianchi identity it must generate the bulk equations of motion.

Bianchi identity: $d\mathcal{P}^I = 0$

Eqns of motion: $d\mathcal{M}_{IJ} {}^*\mathcal{P}^J = 0$

$\longrightarrow$ Self-duality constraint: $\boxed{\mathcal{P}^I = L^{IJ} \mathcal{M}_{JK} {}^*\mathcal{P}^K}$

Require $\mathcal{M} \in O(d,d)/O(d) \times O(d)$ (moduli space of T^d compactif)

$\longrightarrow$ Parameterise in terms of vielbein $\mathcal{V}$: $\mathcal{M} = \mathcal{V}^T \mathcal{V}$

element of O(d,d) identified under left action of O(d) × O(d):
$\mathcal{V} \sim k\,\mathcal{V}$

$\mathcal{V}$ transforms as: $\mathcal{V} \rightarrow k(Y)\, \mathcal{V}\, g$

local O(d) × O(d) transf

rigid O(d,d) transf

$\longrightarrow$ Metric $\mathcal{M} = \mathcal{V}^T \mathcal{V}$:

- Manifestly invariant under local O(d) × O(d) transf

- Transforms under O(d,d) as $\mathcal{M} \rightarrow g^T \mathcal{M}\, g$

From SD-constraint follows: $(L^{IJ}\,\mathcal{M}_{JK})^2 = \mathbb{1}$

In basis where $L = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$:

$$(L^{IJ}\,\mathcal{M}_{JK})^2 = \mathbb{1} \quad \Rightarrow \quad \mathcal{M}_{IJ} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

Can use O(d) × O(d) symmetry to choose triangular gauge for $\mathcal{V}$:

$$\mathcal{V} = \begin{pmatrix} e^T & 0 \\ -e^{-1}B & e^{-1} \end{pmatrix}$$ where $G = e^T e$

Patching

In overlap $U_\alpha \cap U_\beta \subset N$:

$$\mathbb{X}_\alpha = g_{\alpha\beta}^{-1} \, \mathbb{X}_\beta + x_{\alpha\beta}$$

$$\mathcal{P}_\alpha = g_{\alpha\beta}^{-1} \, \mathcal{P}_\beta$$

$$L_\alpha = g_{\alpha\beta}^{-1} \, L_\beta \, g_{\alpha\beta}^{-T}$$

$$\mathcal{M}_\alpha = g_{\alpha\beta}^{T} \, \mathcal{M}_\beta \, g_{\alpha\beta}$$

For $g_{\alpha\beta} \in O(d,d;\mathbb{Z})$: $L_\alpha = L_\beta$ constant metric of signature (d,d)

T-duality: two viewpoints

1. Active

The doubled torus is transformed;
projection onto physical space is kept fixed.

$$\mathcal{M}_\alpha \rightarrow \mathcal{M}_\beta \qquad\qquad R_\alpha = R_\beta$$

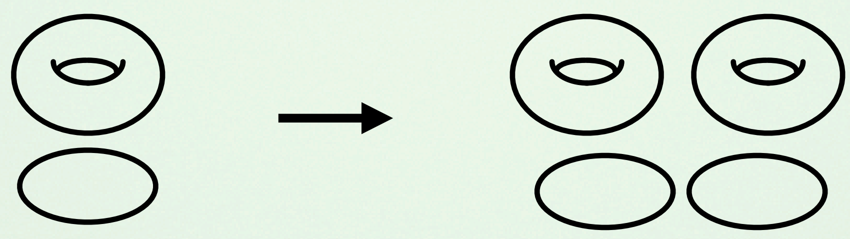
2. Passive

The doubled torus is kept fixed;
projection onto physical space is changed.

$$\mathcal{M}_\alpha = \mathcal{M}_\beta \qquad\qquad R_\alpha \rightarrow R_\beta$$

Doubled geometry

Allow T-duality also along the base $\rightarrow$ double *all* coordinates



May be generalised to other spaces that locally are Lie group manifolds: G/Γ

discrete subgroup $\Gamma \subset G$

Nonlinear sigma model
for open strings on doubled space:

[Hull & Reid-Edwards;
CA, Kimura, Reid-Edwards '08]

$$\boxed{\begin{aligned} S = &\frac{1}{4}\int_\Sigma \mathcal{M}_{MN}\, \mathcal{P}^M \wedge *\mathcal{P}^N \\ &+ \frac{1}{12}\int_V t_{MNP}\, \mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P \\ &- \frac{1}{2}\int_D \omega_{MN}\, \mathcal{P}^M \wedge \mathcal{P}^N \end{aligned}}$$

$\partial V = \Sigma + D$

D = surface on brane

Σ = worldsheet

ω = 2-form restricted by Lie algebra structure

$$\iota \mathcal{T}|_D = \iota\, d\omega$$

$$\mathcal{T} \equiv \frac{1}{12} t_{MNP}\, \mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P$$

$\mathcal{M}_{MN}$ constant

$\mathcal{P}^M = \mathcal{P}^M{}_I\, d\mathbb{X}^I$

$t_{MNP} = t_{MN}{}^Q L_{QP}$ = Lie algebra structure constants on doubled geometry

Bianchi identity: $d\mathcal{P}^M + \frac{1}{2} t_{NP}{}^M \mathcal{P}^N \wedge \mathcal{P}^P = 0$

Eqns of motion:

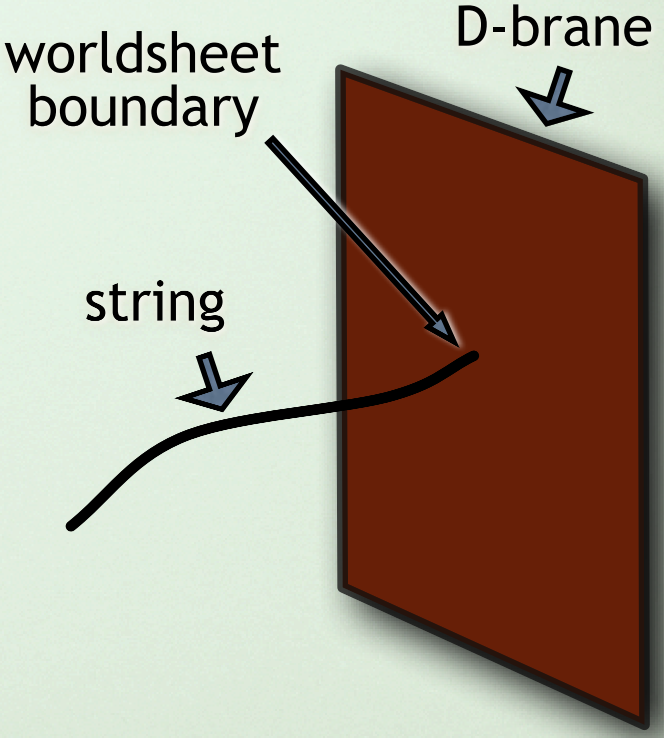
$$d\mathcal{M}_{MN} {}^*\mathcal{P}^N + \mathcal{M}_{NP} t_{MQ}{}^P \mathcal{P}^Q \wedge {}^*\mathcal{P}^N - \frac{1}{2} t_{MNP} \mathcal{P}^N \wedge \mathcal{P}^P = 0$$

$\longrightarrow$ Self-duality constraint: $\boxed{\mathcal{P}^M = L^{MN} \mathcal{M}_{NP} {}^*\mathcal{P}^P}$

D-branes

D-branes = dynamical hypersurfaces to which open strings are attached

→ Described by boundary conditions of nonlinear sigma model



Dirichlet conditions:

conditions on vectors normal to brane

Neumann conditions:

conditions on vectors tangent to brane

Neumann conds
$\partial_n X^i = 0$

T-dual
$\longleftrightarrow$

Dirichlet conds
$\partial_t \tilde{X}_i = 0$

Neumann subspace

Dirichlet subspace

Neumann conds $\partial_n X^i = 0$	$\xleftrightarrow{\text{T-dual}}$	Dirichlet conds $\partial_t \tilde{X}_i = 0$
↙ Neumann subspace		↘ Dirichlet subspace

Introduce Neumann & Dirichlet projectors: Ξ , $\bar{\Xi}$

→ Boundary conditions of doubled sigma model:

$$\begin{cases} \bar{\Xi}^I{}_J \partial_\tau \mathbb{X}^J = 0 & \text{Dirichlet} \\ \Xi^I{}_K \left[-\frac{1}{2} \mathcal{M}_{IJ} \partial_\sigma \mathbb{X}^J + \omega_{IJ} \partial_\tau \mathbb{X}^J \right]_{\partial\Sigma} = 0 & \text{Neumann} \end{cases}$$

Consistency with self-duality constraint

$\Downarrow$

Boundary conditions:

(a)	$\Xi^T L \Xi = \overline{\Xi}^T L \overline{\Xi} = 0$	D-brane is maximally isotropic
(b)	$\Xi^I{}_{I'} \Xi^J{}_{J'} \Xi^K{}_{K'} t_{IJK} = 0$	D-brane is compatible with Lie algebra structure t_{IJK}
(c)	$\Xi^T \mathcal{M} \overline{\Xi} = 0$	normal & tangent vectors on the D-brane are orthogonal w.r.t. $\mathcal{M}$
(d)	$\Xi^I{}_{I'} \Xi^J{}_{J'} \Xi^K{}_{[I,J]} = 0$	integrability

[CA, Kimura, Reid-Edwards ‘08]

Thus in each patch the doubled space has two, in general different, splits into maximally null subspaces:

$$\begin{tikzcd}
 & & T^{2d} \arrow[dl] \arrow[dr] & & \\
\text{D-branes} & (T^d{}_N \,,\, T^d{}_D) \arrow[dr] & & (T^d \,,\, \tilde{T}^d) \arrow[dl] & \text{phys \& dual spaces} \\
 & & T^p{}_{\text{brane}} & &
\end{tikzcd}$$

intersection $\rightarrow$ physical D-brane

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\end{tikzcd}$$

intersection $\rightarrow$ physical D-brane

T-duality changes physical polarisation to different $T^d \subset T^{2d}$

$\longrightarrow$ Intersection with Dirichlet space changes

$\longrightarrow$ Number of physical Dirichlet directions changes

Comparison:
doubled geometry vs generalized geometry

Similarities

★ *Metrics* $L = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$ $\mathcal{M}_{IJ} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \in O(d,d)/O(d)\times O(d)$

★ *Isotropic subspaces:* Subspaces null w.r.t. L are important

★ *Product structure* DG: defines physical & dual spaces

GG: defines split of TM $\oplus$ T*M

★ *T-duality* O(d,d;$\mathbb{Z}$) is naturally incorporated

★ *D-branes:* Intersection of two null subspaces

Differences

- ★ *Doubling* DG: $\dim(M) \to 2\dim(M)$

  GG: $TM \to TM \oplus T^*M$

- ★ *Transition functions* DG: $O(d,d;\mathbb{Z}) \subset GL(2d)$

  GG: $GL(d)$

- ★ *O(n,n) structure* DG: $O(d,d;\mathbb{Z})$

  GG: $O(d,d;\mathbb{R})$

Product structures & complex structures

On doubled geometry we have:

- Product structure $R = \Pi - \tilde{\Pi}$, $R^2 = \mathbb{1}$ (polarisation)
- Product structure $S = L^{-1}\mathcal{M}$, $S^2 = \mathbb{1}$ (from SD-constraint)
- Complex structure $\mathcal{I} = S R$, $\mathcal{I}^2 = -\mathbb{1}$

They satisfy:

$$\mathcal{M}_{IK} S^K{}_J - \mathcal{M}_{JK} S^K{}_I = 0 \qquad \text{compatible}$$

$$L_{IK} S^K{}_J - L_{JK} S^K{}_I = 0 \qquad \text{compatible}$$

$$L_{IK} \mathcal{I}^K{}_J + L_{JK} \mathcal{I}^K{}_I = 0 \qquad \text{hermitian}$$

The structures R, S, $\mathcal{I}$ satisfy a pseudo-quaternionic algebra with sl(2; $\mathbb{R}$) commutation relations:

Define $q_1 = R$, $q_2 = S$, $q_3 = \mathcal{I}$:

$$q_a\, q_b = f_{ab}{}^{c}\, q_c + \eta_{ab}$$

structure constants of sl(2;$\mathbb{R}$): $f_{12}{}^{3} = -1$, $f_{23}{}^{1} = 1$, $f_{31}{}^{2} = 1$

Cartan metric of sl(2;$\mathbb{R}$): $\eta_{ab} = f_{ac}{}^{d}\, f_{bd}{}^{c} = \mathrm{diag}(1,1,-1)$

The structures q_a are preserved by the O(d) diagonal subgroup of O(d) × O(d).

★ Each of the structures R, S, $\mathcal{I}$ defines a generalised complex or real structure for fibres T^d, and together they furnish a *generalised pseudo-hyperkähler structure.*

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Choice of polarisation

$\downarrow$

Choice of maximally isotropic subspace

$\downarrow$

Choice of pure spinor of Spin(d,d) for each point on T^{2d}

$\downarrow$

Pure spinor of Spin(d,d) for each point in T^d fibration

$\downarrow$

Real analogue of generalised CY structure on T^d fibres

(preserved by $GL(d) \subset O(d,d)$ instead of $U(m,m) \subset O(2m,2m)$)

Thank you