Hilbert series and obstructions to asymptotic Chow semistability

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Supersymmetry in complex geometry

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Talk based on a joint work with Hajime Ono and Yuji Sano.

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(or the space of Kähler forms).

We seek a constant scalar curvature Kähler (cscK) metric in $c_1(L)$.

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Theorem A. (Donaldson, JDG 2001)
Let (M, L) be a polarized manifold with $\mathrm{Aut}(M, L)$ discrete.
If $\exists$ a cscK metric
then (M, L) is asymptotically Chow stable.

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$d_k =$ the degree of M_k in $\mathbb{P}(V_k)$

Let $m = \dim_{\mathbb{C}} M$.

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and this divisor is defined by

$\widehat{M}_k \in (\mathrm{Sym}^{d_k}(V_k))^{\otimes(m+1)}$.

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Stabilizer of $\widehat{M}_k$ under $SL(V_k)$-action is $\mathrm{Aut}(M, L)$.

In Theorem A, "$\mathrm{Aut}(M, L)$ is discrete" means "the stabilizer is finite".

M is said to be <u>Chow polystable</u> w.r.t. L^k if the orbit of $\widehat{M}_k$ in $(\mathrm{Sym}^{d_k}(V_k))^{\otimes(m+1)}$ under the action of $\mathrm{SL}(V_k)$ is <u>closed</u>.

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M is <u>Chow stable</u> w.r.t L^k if M is <u>polystable</u> and the <u>stabilizer</u> at $\widehat{M}_k$ of the action of $\mathrm{SL}(V_k)$ is <u>finite</u>.

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M is <u>Chow semistable</u> w.r.t. L^k if the <u>closure of the orbit</u> of $\widehat{M}_k$ in $(\mathrm{Sym}^{d_k}(V_k))^{\otimes(m+1)}$ under the action of $\mathrm{SL}(V_k)$ <u>does not</u> <u>contain</u> $\mathbf{o} \in (\mathrm{Sym}^{d_k}(V_k))^{\otimes(m+1)}$.

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M is Chow stable w.r.t L^k if M is polystable and the stabilizer at $\widehat{M}_k$ of the action of $\mathrm{SL}(V_k)$ is finite.

M is Chow semistable w.r.t. L^k if the closure of the orbit of $\widehat{M}_k$ in $(\mathrm{Sym}^{d_k}(V_k))^{\otimes(m+1)}$ under the action of $\mathrm{SL}(V_k)$ does not contain $\mathbf{o} \in (\mathrm{Sym}^{d_k}(V_k))^{\otimes(m+1)}$.

M is asymptotically Chow polystable (resp. stable or semistable) w.r.t. L if there exists a $k_0 > 0$ such that M is Chow polystable (resp. stable or semistable) w.r.t. L^k for all $k \geq k_0$.

Theorem B (Mabuchi, Osaka J. Math. 2004)

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If Aut(M, L) is not discrete then

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Theorem C (Mabuchi, Invent. Math. 2005)

Let (M, L) be a polarized manifold, and suppose Aut(M, L) is not discrete.

If $\exists$ a cscK metric in $c_1(L)$ and if the obstruction vanishes then (M, L) is asymptotically Chow polystable.

$$
\begin{aligned}
\mathfrak{h}_0 &= \{X \text{ holo vector field} \mid \text{zero}(X) \neq \emptyset\} \\
&= \{X \text{ holo vector field} \mid \exists u \in C^\infty(M) \otimes \mathbb{C} \\
&\qquad\qquad \text{s.t. } X = \text{grad}' u = g^{i\bar{j}} \frac{\partial u}{\partial \overline{z}^j} \frac{\partial}{\partial z^i}\} \\
&= \{X \text{ holo vector field} \mid \exists \text{lift to an infinitesimal action on } L\} \\
&= \text{Lie}(\text{Aut}(M, L))
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Theorem D (F, Internat. J. Math. 2004)

Let (M, L) be a polarized manifold with $\dim_{\mathbb{C}} M = m$.

(1) The vanishing of Mabuchi's obstruction is equivalent to the vanishing of Lie algebra characters $\mathcal{F}_{\mathrm{Td}^i} : \mathfrak{h}_0 \to \mathbb{C}$, $i = 1, \cdots, m$.

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(2) $\mathcal{F}_{\mathrm{Td}^1}$ = obstruction to $\exists$ of cscK metric (Futaki invariant).

Question (a) In Theorem C, can't we omit the assumption of the vanishing of the obstruction ?

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Question (b) In Theorem D, if $\mathcal{F}_{\mathrm{Td}^1} = 0$ then $\mathcal{F}_{\mathrm{Td}^2} = \cdots = \mathcal{F}_{\mathrm{Td}^m} = 0$?

Question (c) $\dim \mathrm{span}\{\mathcal{F}_{\mathrm{Td}^1}, \cdots, \mathcal{F}_{\mathrm{Td}^m}\} = 1$?

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(3) The derivatives of the Hilbert series are computed by imputting toric data into a computer.

Remark: Questions (a) and (b) are still open.

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Then the lift $\widetilde{X}$ to L of X is written as

$$\widetilde{X} = \theta(\widetilde{X}) iz \frac{\partial}{\partial z} + X^\sharp.$$

The ambiguity of $\widetilde{X}$ is const $iz\frac{\partial}{\partial z}$.

$$\begin{aligned}
\widetilde{X} + ciz\frac{\partial}{\partial z} &= \theta(\widetilde{X} + ciz\frac{\partial}{\partial z})iz\frac{\partial}{\partial z} + X^{\sharp} \\
&= (\theta(\widetilde{X}) + c)iz\frac{\partial}{\partial z} + X^{\sharp}.
\end{aligned}$$

Given a Kähler form $\omega \in c_1(L)$, suppose the connection θ is so chosen that

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Conclusion: **Ambiguity of Hamiltonian function $\Longleftrightarrow$ ambiguity of lifting of X to L.**

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Put

$$L(X) = \nabla_X - L_X \in \Gamma(\mathrm{End}(T'M))$$

and let

$$\Theta \in \Gamma(\Omega^{1,1}(M) \otimes \mathrm{End}(T'M))$$

be the (1,1)-part of the curvature form of ∇.

Def: For $\phi \in I^p(GL(m,\mathbb{C}))$, we define

$$\mathcal{F}_\phi(X) = (m-p+1)\int_M \phi(\Theta)\wedge u_X\,\omega^{m-p} + \int_M \phi(L(X)+\Theta)\wedge\omega^{m-p+1}. \tag{1}$$

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Theorem D : Vanishing of Mabuchi's obstruction is equivalent to

$$\mathcal{F}_{\mathrm{Td}^1}(X) = \cdots = \mathcal{F}_{\mathrm{Td}^m}(X) = 0$$

for all $X \in \mathfrak{h}_0$.

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For $g \in T^{m+1}$,

$$L(g) := \sum_{k=0}^{\infty} \mathrm{Tr}(g|_{H^0(M, K_M^{-k})})$$

the formal sum of the Lefchetz numbers.

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Let $\{v_j \in \mathbb{Z}^m\}_j$ be the generators of 1-dimensional faces of the fan of M.

$\lambda_j := (v_j, 1) \in \mathbb{Z}^{m+1}$,
$C^* := \{y \in \mathbb{R}^{m+1} | \lambda_j \cdot y \geq 0, \forall j\} \subset \mathfrak{g}^* = (\mathrm{Lie}(T^{m+1}))^*$,
$P^* := \{w \in \mathbb{R}^m | v_j \cdot w \geq -1, \forall j\}$

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C^* is a cone over P^*.

Integral points in $C^* \longleftrightarrow \cup_{k=1}^{\infty}$ basis of $H^0(M, K_M^{-k})$

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Def: $\mathcal{C}(\mathbf{x}, C^*) := \sum_{\mathbf{a} \in C^* \cap \mathbb{Z}^{m+1}} \mathbf{x}^{\mathbf{a}}$ **Hilbert series**

Fact: $\mathcal{C}(\mathbf{x}, C^*)$ **is a rational function of** $\mathbf{x}$**.**

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For $\mathbf{b} \in \mathbb{R}^{m+1} \cong \mathfrak{g} = \mathrm{Lie}(T^{m+1})$**,**

$e^{-t\mathbf{b}} := (e^{-b_1 t}, \cdots, e^{-b_{m+1} t})$

$\mathcal{C}(e^{-t\mathbf{b}}, C^*) = \sum_{\mathbf{a} \in C^* \cap \mathbb{Z}^{m+1}} e^{-t\mathbf{a} \cdot \mathbf{b}}$ **: rational function in** t

$C_R := \{(b_1, \cdots, b_m, m+1) | (b_1, \cdots, b_m) \in (m+1)P\} \subset \mathfrak{g}$

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P is the dual polytope of P^*.

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C_R is the space of Reeb vector fields of Sasakian structures on S, the total space of the associated $U(1)$-bundle of K_M.

The tangent space of C_R at $(0, \cdots, 0, m+1)$ is
$T_{(0,\cdots,0,m+1)} C_R = \{\mathbf{c} = (c_1, \cdots, c_m, 0)\} \subset \mathfrak{g}$.
This defines another way of lifting of T^m-action to L.

Put $\mathbf{b} = (0, \cdots, 0, m+1)$.

Comparing the two liftings of T^m-action to L we can show

Theorem: (1) The coeffocients of the Laurant series of the rational function $\frac{d}{ds}|_{s=0}\mathcal{C}(e^{-t(\mathbf{b}+s\mathbf{c})}, C^*)$ in t span the linear space spanned by $\mathcal{F}_{\mathrm{Td}^1}, \cdots, \mathcal{F}_{\mathrm{Td}^m}$.

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(2) In dimension 2, the linear spans are 1-dimensional.
In dimension 3 the linear spans are at most 2-dimensional, and there are examples in which the linear spans are 2-dimensional.

Remark Martelli-Sparks-Yau: From t^{-m} term we get the Futaki invariant.

A question we can not answer is whether or not

there is a polarized manifold (M, L)

on which a cscK metric exists in $c_1(L)$

so that $\mathcal{F}_{\mathrm{Td}^1} = 0$

but on which $\mathcal{F}_{\mathrm{Td}^p} \neq 0$ for some $p = 2, \cdots, m$.

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If the answer is no, the assumption on

the obstruction in Mabuchi's result can be omitted.

Our computations show that the last question is closely related to a question raised by Batyrev and Selivanova:

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Is a toric Fano manifold with vanishing $f(=\mathcal{F}_{\mathrm{Td}^1})$ for the anticanonical class necessarily symmetric?

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Is a toric Fano manifold with vanishing $f(= \mathcal{F}_{\mathsf{Td}^1})$ for the anticanonical class necessarily symmetric?

If the answer is yes, then any toric Fano Kähler-Einstein manifold has vanishing $\mathcal{F}_{\mathsf{Td}^p}$ for $p = 1, \cdots, m$.

Recall that a toric Fano manifold M is said to be symmetric if the trivial character is the only fixed point of the action of the Weyl group on the space of all algebraic characters of the maximal torus in Aut(M).

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Note that if a toric Fano manifold M is symmetric then the character f for the anticanonical class vanishes.

Recall also that Batyrev and Selivanova proved that a toric Fano manifold M admits a Kähler-Einstein metric if M is symmetric,

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and that Wang and Zhu improved the result of
Batyrev and Selivanov to the effect that
a toric Fano manifold M admits a Kähler-Einstein
metric if the invariant f vanishes
for the anticanonical class.