Holomorphic Poisson structures and deformations of generalized Kähler structures II

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Theorem 0.1 $(\mathcal{J}, \psi)$: G. Kähler structures with one pure spinor on a compact manifold X. Let $\{\mathcal{J}_t\}_{t\in D}$ be an analytic family of G. cpx strs , parametrized by the complex disk D of 1 dim. containing the origin 0, with $\mathcal{J}_0 = \mathcal{J}$. Then there exists a family of G. Kähler strs with one pure spinor $(\mathcal{J}_t, \psi_t)_{t\in D'}$, parametrized by a small disk D' containing the origin, with $\psi_0 = \psi$.

Theorem 0.1 implies that G. Kähler strs with one pure spinor are stable under small deformations of G. cpx strs, which is a generalization of the stability theorem of Kodaira-Spencer, Known results on bi-Hermitian and generalized Kähler structures

- approaches by the reduction
  (Y. Lin and S. Tolman [LT], Bursztyn, Gualtieri and Cavalcanti [BGC])

- A construction of bi-Hermitain str. on Fano surfaces (Hitchin) by Hamiltonian diffeomorphisms

$$\{(J_+(t), J_-)\}$$

Higher dimensional generalization to Poisson manifolds (Gualtieri)
they are restricted deformations such that complex structures do not change, $(X, J_+(t)) = (X, J_-) = (X, J)$

- blown up Hopf surfaces (LeBrun)
- hyperbolic Inoue surfaces by using twistor spaces (Fujiki-Pontecorvo)

Let (X, J, ω) be a compact Kähler manifold with a holomorphic Poisson structure β. Then there is a family of bi-Hermitain structures $(J_+(t), J_-(t))$

$\beta\cdot\omega \in \Theta \otimes \wedge^{0,1}$: contraction of β by ω

$[\pm\beta\cdot\omega] \in H^1(X, \Theta)$: Kodaira-Spencer classes of deformations of bi-Hermitain structures $\{J_\pm(t)\}$.

$$J_+(t) = J + \beta \cdot \omega t + O(t^2), \quad J_-(t) = J - \beta \cdot \omega t + O(t^2)$$

Theorem 0.2 The obstruction class of the infinitesimal deformation $[\beta \cdot \omega]$ vanishes, i.e.,

$$[\beta \cdot \omega,\, \beta \cdot \omega] = 0 \in H^2(X, \Theta)$$

For instance $(X, J) = \mathbf{C}P^1 \times T^n_{\mathbf{C}}$ $(n > 1)$. There are obstructions to deformations of complex structures on (X, J). However the deformations given by holomorphic Poisson structures are unobstructed.

1 Holomorphic Poisson structures

X: compact Kähler manifold with Kähler form ω

$\beta \in H^0(X, \wedge^2 \Theta)$: holomorphic 2-vector filed

β: holomorphic Poisson structure

$\Leftrightarrow [\beta, \beta]_{\mathrm{Scou}} = 0$ (Scouten bracket)

A holomorphic Poisson str. β gives deformations of G.complex strs by $\mathcal{J}_{\beta t} := \mathrm{Ad}_{e^{\beta t}} \mathcal{J}_J$.

From theorem 0.1 (the stability theorem),

Theorem 1.1 A holomorphic Poisson structure β gives rise to non-trivial deformations of G.Kähler structures $(\mathcal{J}_{\beta t}, \psi_t)$.

rank β_x: the rank of 2-vector β_x, $x \in X$

$$\mathrm{Type}\,(\mathcal{J}_\beta)_x = n - 2\,\mathrm{rank}\,\beta_x$$

From theorem 1.1 every compact Toric Kähler manifolds admits non-trivial G. Kähler structures, (at least 2 dim).
$\{V_i\}_{i=1}^{l}$: l commuting holomorphic vector fields
Then

$$\beta = \sum_{i,j} \lambda_{i,j} V_i \wedge V_j,$$

gives a holomorphic Poisson structure where each $\lambda_{i,j}$ is a constant.

2 G. Kähler strs. on Fano surfaces

$X = \mathbf{C}P^2$, $\wedge^2\Theta \cong K^{-1} \cong \mathcal{O}(3)$

$\beta \in H^0(\mathbf{C}P^2, \wedge^2\Theta) \cong H^0(\mathbf{C}P^2, \mathcal{O}(3))$

the zero set of β is the cubic curve of $\mathbf{C}P^2$

Theorem 0.1 implies a hol. 2-vector β generates deformations of G. Kähler strs on $\mathbf{C}P^2$.

S_n : blown up $\mathbf{C}P^2$ at generic n points

$(n \leq 8)$

$H^1(S_n, \Theta)$:infinitesimal deformations of cpx strs of S_n

Deformations of G. cpx strs are given by

$H^0(S_n, K^{-1}) \oplus H^1(S_n, \Theta)$.

No obstructions to deformations.

$$\dim H^1(S_n, \Theta) = \begin{cases} 2n-8, & (n = 4, \cdots, 8) \\ 0, & (n = 0, 1, 2, 3) \end{cases}$$

$$\dim H^0(S_n, K^{-1}) = 10 - n$$

Deformations of G. Kähler strs are given by

$$H^0(S_n, K^{-1}) \oplus H^1(S_n, \Theta) \oplus H^{1,1}(S_n).$$

$$H^{1,1}(S_n) = 1 + n$$

3 An applicatoin to bi-Hermitian geometry

(X, J_+, J_-, h) : bi-Hermitian sturucutre

$\Leftrightarrow J_+, J_-$: cpx structures ,

h : Hermitain metric w.r.t. both J_+ and J_-

Torsion condition:

$$-d^c_+\omega_+ = d^c_-\omega_- = db,$$

where $d^c_\pm = \sqrt{-1}(\overline{\partial}_\pm - \partial_\pm)$,

$\overline{\partial}_\pm$: $\overline{\partial}$ operator w.r.t $J_\pm$.

Theorem 3.1 (Gualitieri) G. Kähler str $\Leftrightarrow$ bi-Hermitain str with the torsion condition

From theorem 0.1 and 3.1,

Theorem 3.2 S_n : blown up $\mathbf{C}P^2$ at $n(\leq 8)$ points. Then there exists a family of bi-Hermitain structures on S_n which is parametrized by $H^0(S_n, K^{-1}) \oplus H^1(S_n, \Theta) \oplus H^{1,1}(S_n)$

4 bi-Hermitain structures on ruled surfaces

$(X, J) \to \Sigma$: minimal ruled surface over a Riemannian surface Σ with genus g. If there is a non-trivial holomorphic Poisson strucutre on X, then there exists a family of bi-Hermitian structures $(J_+(t), J_-(t))$ on X.

$$J_+(0) = J_-(0) = J$$

Fact: small deformations of (X, J) are still ruled surfaces for genus $g \geq 1$.

Thus $(X, J_\pm(t))$ is a ruled surface w.r.t both $J_\pm(t)$.

for instance, $\mathbb{P}(T^*\Sigma \oplus \mathcal{O}_\Sigma) \to \Sigma$ admits a Poisson structure.

$(X, J) = \mathbb{F}_2 = \mathbb{P}(T^*\mathbf{C}P^1 \oplus \mathcal{O}_{\mathbf{C}P^1})$: ruled surfaces

deformations of $\mathbb{F}_2$ can be the product $\mathbf{C}P^1 \times \mathbf{C}P^1$.

Applying our theorem of deformations, we obtain a family of bihermitian structures $(J_+(t), J)$ such that $(X, J_+(t)) = \mathbf{C}P^1 \times \mathbf{C}P^1$ $(t \neq 0)$ and $(X, J) = \mathbb{F}_2$.

It is because that there is no obstruction to deformations ogeneralized complex structures on $\mathbb{F}_2$, i.e., $H^2(\mathbb{F}_2, \Theta) \oplus H^1(\mathbb{F}_2, \wedge^2\Theta) = \{0\}$, and the map $H^0(\mathbb{F}_2, \wedge^2\Theta) \times H^{1,1}(\mathbb{F}_2) \to H^1(\mathbb{F}_2, \Theta)$ is surjective.

5 idea of proof

1. generalized Hodge decomposition of generalized Kähler manifolds

2. To show that the obstruction classes to deformations vanish. An analogous to the method in the theorem of unobstructed deformations of Calabi-Yau manifolds due to Bogomolov-Tian-Todorov.

3. Obstruction space does not vanish in general, however the obstruction class vanishes. The method is modified to

apply to the cases of generalized Kähler deformations

4. Key Point is to use non-degenerate pure spinors which are closed differential forms.

6 generalized Hodge decomposition

$(X, \mathcal{J}_1)$: generalized complex manifold of $\dim_{\mathbf{R}} = 2n$.

$\mathcal{J} \in$SO$(T \oplus T^*) \subset$ CL

CL acts on $\wedge^* T^*$ by the Spin representation. Then $\wedge^* T^*$ is decomposed into eigenspaces

$$\wedge^* T^* = U_{\mathcal{J}_1}^{-n} \oplus U_{\mathcal{J}_1}^{-n+1} \oplus \cdots \oplus U_{\mathcal{J}_1}^{n}$$

Since $\mathcal{J}_1$ is integrable, the exterior derivative is decomposed, $d = \partial_{\mathcal{J}_1} + \overline{\partial}_{\mathcal{J}_1}$

$$\partial_{\mathcal{J}_1} : U_{\mathcal{J}_1}^{p} \to U_{\mathcal{J}_1}^{p-1}, \quad \overline{\partial}_{\mathcal{J}_1} : U_{\mathcal{J}_1}^{p} \to U_{\mathcal{J}_1}^{p+1}$$

$(X, \mathcal{J}_1, \mathcal{J}_2)$: generalized Kähler manifold

$[\mathcal{J}_1, \mathcal{J}_2] = 0$, We have the simultaneous decomposition,

$$U^{p,q} = U^p_{\mathcal{J}_1} \cap U^q_{\mathcal{J}_2}$$

$$\wedge^* T^* = \bigoplus_{p.q} U^{p,q}$$

There is a generalized Hodge star operator $*$ which induces a volume form and a metric on $\wedge^* T^*$.

$$\mathrm{Vol}_G = *1, \quad G(\alpha, \beta) = \langle \alpha, *\beta \rangle \mathrm{Vol}_G$$

The exterior derivative d is decomposed

$$d = \delta_+ + \delta_- + \overline{\delta_+} + \overline{\delta_-}$$

As in Kähler manifolds, we have

$$\triangle_d = 2\triangle_{\partial_{\mathcal{J}_i}} = 2\triangle_{\overline{\partial}_{\mathcal{J}_i}} = 4\triangle_{\delta_\pm} = 4\triangle_{\overline{\delta}_\pm}$$

Thus we have the Harmonic theory,

Theorem 6.1 (Gualtieri)

$$H^{even/odd}(X) = \bigoplus_{p+q=even/odd} \mathbb{H}^{p,q}(X),$$

where

$$\mathbb{H}^{p,q}(X) = \ker \triangle_d \cap U^{p,q}$$

7 sketch of proof

Given deformations of G.cpx structures $\mathcal{J}_t$, there exists a family of sections $\exists a(t) \in \mathrm{CL}^2$ such that

$$\mathrm{Ad}_{e^{a(t)}} \mathcal{J}_0 = \mathcal{J}_t.$$

$$a(t) = a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \cdots$$

Then we shall construct a formal power series $b(t)$ satisfying :

$$d(e^{a(t)}e^{b(t)}\psi) = 0, \qquad \text{(eq)}$$
$$\mathrm{Ad}_{e^{b}(t)}\mathcal{J} = \mathcal{J}$$

From the Campbell-Hausdorff formula,

$$e^{z(t)} = e^{a(t)}e^{b(t)}, \quad (z(t) \in \mathrm{CL}^2).$$

If we obtain such a $b(t)$, then we have the action of Clifford group $e^{z(t)}$ on $\mathcal{J}$,

$$\mathrm{Ad}_{e^{z(t)}}\mathcal{J} = \mathrm{Ad}_{e^{a(t)}}\mathrm{Ad}_{e^{b(t)}}\mathcal{J} = \mathrm{Ad}_{e^{a(t)}}\mathcal{J} = \mathcal{J}_t,$$

which preserves deformations of G. complex str $\{\mathcal{J}_t\}$. Put $\psi_t = e^{z(t)}\psi$. Then

$$de^{z(t)}\psi = de^{a(t)}e^{b(t)}\psi = 0.$$

Thus the pair $(\mathcal{J}_t, \psi_t)$ is a G. Kähler str. We determine b_k inductively,

$$b(t) = b_1 t + b_2 \frac{t^2}{2!} + \cdots .$$

We solve the equation $de^{a(t)}e^{b(t)}\psi = 0$. the k-th order term in t of (eq) is given by

$$\frac{1}{k!}db_k\psi + \mathrm{Ob}_k(a_{<k}, b_{<k}) = 0.$$

We introduce a differential graded module over CL,

$$(K^*, d)$$

$$K^1 = U^{0,-n+2}$$

$$K^2 = U^{1,-n+1} \oplus U^{-1,-n+1} \oplus U^{1,-n+3} \oplus U^{-1,n+3}$$

Then we have an elliptic complex,

$$0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \xrightarrow{d} \cdots$$

with the (finite dimensional) cohomology groups $H^i(K^*)$.

(K^*, d) is a subcomplex of the full de Rham complex.

From the generalized Hodge decomposition of G. Kähler manifolds, it turns out that the map $p^i : H^i(K^*) \to H^*_{\mathrm{dR}}(X)$ is injective.

$\mathrm{Ob}_k = \mathrm{Ob}_k(a_{<k}, b_{<k}) \in K^2$ defines

$[\mathrm{Ob}_k] \in H^2(K^*)$: the obstruction class

$\frac{1}{k!} d b_k \psi \in d\Gamma(K^1)$.

$$\frac{1}{k!} d b_k \psi + \mathrm{Ob}_k(a_{<k}, b_{<k}) = 0.$$

$$0 \longrightarrow K^1 \xrightarrow{d} K^2 \longrightarrow \cdots$$

$$b_k \psi \mapsto -(k!)\mathrm{Ob}_k$$

Ob_k is a d-exact form in K^2. Since the map p^2 is injective, the obstruction class $[\mathrm{Ob}_k]$ vanishes. Hence there exists a solution b_k for each k and we obtain a formal power series $b(t)$ as a solution of the equation (eq). Finally we show that the formal power series $b(t)$ converges.

8 Generalized Kähler submanifolds

Definition 8.1 $(X^{2n}, \mathcal{J}, \psi)$ G.Kähler manifold with one pure spinor. A submanifold $i_M : M^{2m} \to (X^{2n}, \mathcal{J}, \psi)$ admits $\mathcal{J}$-invariant conormal bundle if the conormal bundle $N^*_{M|X}$ to M in X satisfies

$$\mathcal{J}(N^*_{M|X}) = N^*_{M|X}.$$

Rremark If a submanifold M admits $\mathcal{J}$-invariant conormal bundle, M is a generalized complex submanifold in the sense of O. Ben-Bassart and M. Boyarchenko.

Then $\mathcal{J}$ induces the G.almost complex str. $\mathcal{J}_M$ and we have

Theorem 8.2 $\mathcal{J}_M$ is integrable.

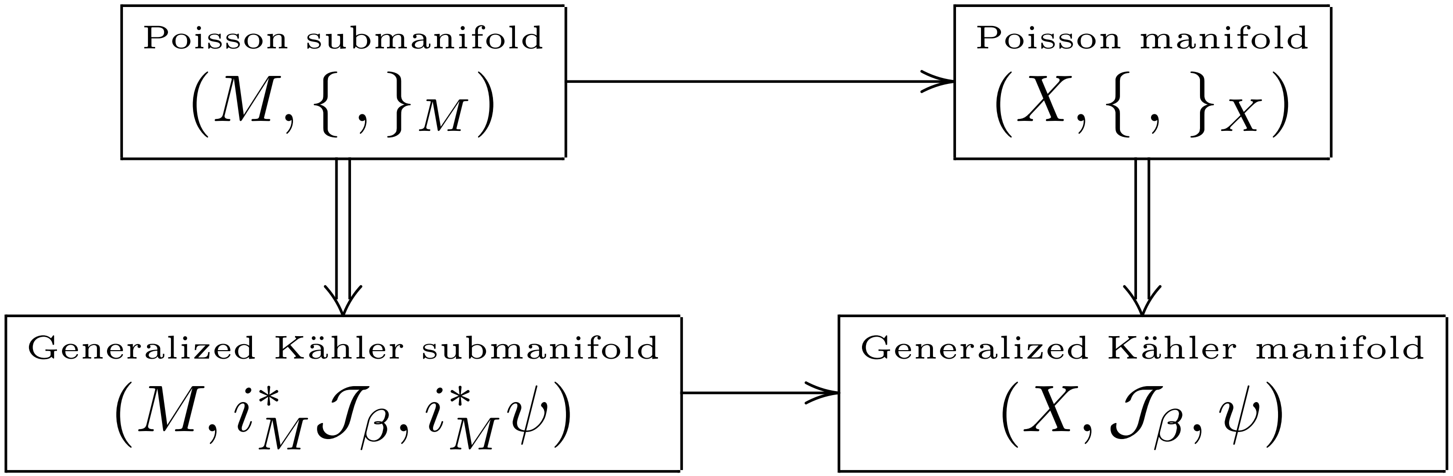
Theorem 8.3 $i_M^* \psi$ is a non-degenerate, pure spinor on M. In particular, $(M, \mathcal{J}_M, i_M^*\psi)$ is a G.Kähler submanifold with one pure spinor.

X : compact Kähler maniofold with hol. Poisson structure β. ($\beta \Leftrightarrow$ the Poisson structure $\{\, , \,\}_X$).

Definition 8.4 Let M be a complex submanifold defined by an ideal I_M. A complex submanifold M of X is a Poisson submanifold if $\beta(df) \subset I_M \otimes TX$, for all $f \in I_M$. This is equivalent to say that the Poisson bracket $\{\, ,\, \}_X$ induces a Poisson bracket $\{\, ,\, \}_M$ on M.

We have a family of G.Kähler structures $(\mathcal{J}_{\beta,t}, \psi_t)$ from theorem 0.1. Then

Theorem 8.5 Let $i_M : M \to (X, \beta)$ be a Poisson submanifold. Then M is a generalized Kähler submanifold of $(X, \mathcal{J}_{\beta,t}, \psi_t)$.



Many interesting examples of G.Kähler submanifolds arise as Poisson submanifolds.