Holomorphic Poisson structures and deformations of generalized Kähler structures

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R.Goto, On deformations of generalized Calabi-Yau, HyperKähler, G_2 and Spin(7) structures, math.DG/051221

R. Goto, Deformations of generalized complex and generalized Kähler structures, math.DG/0705.2495

R. Goto, Poisson structures and generalized Kähler structures, math.D.G/0712.2685

X : compact real manifold of $2n$ dim

$T := TX$: tangent bundle of X

$T^* := T^*X$: cotangent bundle of X

1 Introduction

A notion of generalized complex structures are introduced by Hitchin which interpolates symplectic structures and complex structures.

A generalized Kähler structure is a generalization of ordinary Kähler structure which inherits remarkable properties of Kähler geometry such as the Hodge decomposition, (generalized Hodge decomposition)

Theorem (Kodaira-Spencer) Small deformations of Kähler manifolds are still Kählerian
(stability theorem of Kähler structures)

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Ans. Yes,
"generalized Kähler structures with one pure spinor" are stable under small deformations of generalized complex structures.

Question: Are there nontrivial generalized Kähler structure on holomorphic Poisson Kähler manifolds ?

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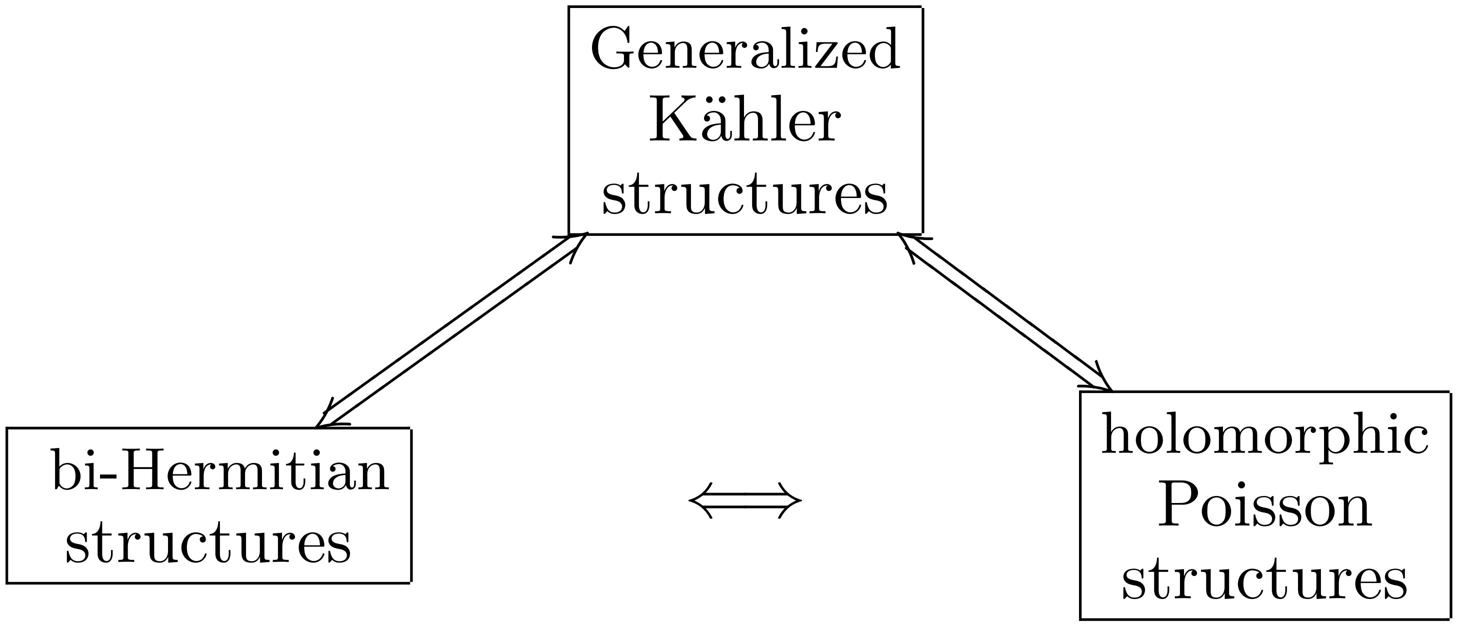
Ans. Yes !


Generalized Kähler structures
bi-Hermitian structures
$\Longleftrightarrow$
holomorphic Poisson structures

Generalized geometry

idea : To replace the tangent bundle T on a manifold with the direct sum $T \oplus T^*$!

$$T \Rightarrow T \oplus T^*$$

$\langle\, ,\, \rangle$: metric on $T \oplus T^*$ with sgn $(2n, 2n)$

$\mathrm{O}(T \oplus T^*)$: the orthogonal group bundle,

$\mathrm{SO}(T \oplus T^*)$: special orthogonal group bundle.

Clifford algebra CL of $T \oplus T^*$

Clifford group G_{cl}

geometric structures with the symmetry of

Clifford group $G_{\mathrm{cl}} \to$ (almost) generalized geometric str.

$$1 \to \underline{\mathbf{R}}^* \to G_{\mathrm{cl}} \to O(T \oplus T^*) \to 1$$
$$\mathrm{CL}^0 \subset \mathrm{CL}^2 \subset \cdots \subset \mathrm{CL}$$

CL^2 : the Lie algebra of the Clifford group G_{cl}

Then small deformations of almost generalized geometric structures are given by

$$\Phi_t = e^{a(t)} \cdot \Phi, \quad (a(t) \in \mathrm{CL}^2[[t]])$$

$$\mathrm{CL}^2 = C^\infty(X) \oplus \mathrm{End}(\mathrm{T}) \oplus \wedge^2 T^* \oplus \wedge^2 T$$

End(T)$\Leftrightarrow$ ordinary deformations of structures

$b \in \wedge^2 T^* \Leftrightarrow$: transformations by b-field

$\beta \in \wedge^2 T \Leftrightarrow$ transformations by 2-vector,

(Poisson deformations)

We need to solve a PDF to obtain integrable generalized structures.

An almost generalized complex structure on X is a section $\mathcal{J} \in \mathrm{SO}(T \oplus T^*)$ with

$$\mathcal{J}^2 = -\mathrm{id}.$$

$$\mathcal{J} \; + \text{an integrability condition}$$
$$\Downarrow$$
$$\text{generalized complex structure.}$$

The eigenspace decomposition of $(T \oplus T^*) \otimes \mathbf{C}$ w.r.t. $\mathcal{J}$

$$(T \oplus T^*) \otimes \mathbf{C} = L \oplus \overline{L}$$

$\mathcal{J}$: integrable $\Leftrightarrow L$ is involutive

w.r.t. the Courant bracket.

The set of almost G. cpx strs forms an orbit of the adjoint action of $\mathrm{O}(T \oplus T^*)$ on $\mathrm{End}(T \oplus T^*)$.

J : (ordinary) complex str. on X

J^* : the cpx str. on T^*

generalized complex structure $\mathcal{J}_J$ is given by

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$$

ω : symplectic str. on X

generalized complex structure $\mathcal{J}_\omega$ is given by

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

($\omega : T \to T^*$, $\omega^{-1} : T^* \to T$.)

Thus a cpx str.J and a symplectic str. ω on a mfd X respectively gives rise to generalized complex structures $\mathcal{J}_J$ and $\mathcal{J}_\omega$. there exists a section $g \in \mathrm{O}(T \oplus T^*)$ such that $\mathrm{Ad}_g \mathcal{J}_J = \mathcal{J}_\omega$ (locally).

$\mathcal{J}$: G. cpx str. on a manifold X. For each point $x \in X$, we have the type number,

$$\mathrm{Type}\, \mathcal{J}_x \in \{0, 1, \cdots, n\}.$$

type 0 $\Rightarrow$ symplectic structures ($+b$-fields)

type 1

type 2 intermediate structures

$\vdots$

type n $\Rightarrow$ complex structures($+b$-fields)

generalized cpx strs interpolate between complex strs and symlectic strs.

What are intermediate structures ?

$\mathcal{J}$ on X with a constant type number l

Then $\mathcal{J}$ defines a foliation on X with transversally holomorphic str. of cpx l dim and leafwise symplectic structure.

However

In general, type $\mathcal{J}$ may jump from a point to a point

2 Pure spinors

$E = v + \theta \in T \oplus T^*$ acts on a differential form $\psi \in \wedge^* T^*$ by the interior product i_v and the exterior product $\theta\wedge$, $(v \in T, \theta \in T^*)$

$$E \cdot \psi := i_v \psi + \theta \wedge \psi$$

Then

$$E \cdot E \cdot \psi = 2\langle E, E\rangle \psi.$$

Thus we have the induced action of the Clifford algebra $\mathrm{CL}(T \oplus T^*)$ on $\wedge^* T^*$

(Spin representaion).

isotropic subspace L_ψ

$$L_\psi := \{\, E \in (T \oplus T^*) \otimes \mathbf{C} \,|\, E \cdot \psi = 0 \,\}$$

$\psi \in \wedge^* T^*$ is a (complex) pure spinor $\overset{def}{\Leftrightarrow}$

$$\dim_{\mathbf{C}} L_\psi = 2n \, (\text{maximal dim.})$$

pure spinor ψ is nondegenerate if we have

(1) $$(T \oplus T^*) \otimes \mathbf{C} = L_\psi \oplus \overline{L}_\psi .$$

This is equivalent to non-vanishing of spinor norm of ψ,

$$\langle \psi, \overline{\psi} \rangle \neq 0$$

A nondegenerate, pure spinor ψ gives the decomposition (1) which induces the
almost generalized complex str $\mathcal{J}_\psi$.

$$\psi \Rightarrow \mathcal{J}_\psi$$

If $d\psi = 0$, the corresponding G. cpx str $\mathcal{J}_\psi$ becomes integrable.

3 Generalized Kähler structures

Definition 3.1 A pair $(\mathcal{J}_0, \mathcal{J}_1)$ consisting of commuting G. cpx strs is a generalized Kähler structure if the composition

$$\hat{G} := -\mathcal{J}_0\mathcal{J}_1 = -\mathcal{J}_1\mathcal{J}_0 \in \mathrm{SO}(T \oplus T^*)$$

gives a positive definite metric, i.e.,
$G(E_1, E_2) := \langle \hat{G}E_1, E_2\rangle$. $(E_1, E_2 \in T \oplus T^*.)$

$\mathcal{J}_0$: G. cpx. str.

ψ : nondegenerate, pure spinor with $d\psi = 0$

Definition 3.2 $(\mathcal{J}_0, \psi)$:*generalized Kähler structures with one pure spinor* $\Leftrightarrow$ the induced pair G. cpx str $(\mathcal{J}_0, \mathcal{J}_\psi)$ is a G. Kähler str.

Theorem 3.3 $(\mathcal{J}, \psi)$: G. Kähler structures with one pure spinor on a compact manifold X. Let $\{\mathcal{J}_t\}_{t\in D}$ be an analytic family of G. cpx strs , parametrized by the complex disk D of 1 dim. containing the origin 0, with $\mathcal{J}_0 = \mathcal{J}$. Then there exists a family of G. Kähler strs with one pure spinor $(\mathcal{J}_t, \psi_t)_{t\in D'}$, parametrized by a small disk D' containing the origin, with $\psi_0 = \psi$.

Theorem 3.3 implies that G. Kähler strs with one pure spinor are stable under small deformations of G. cpx strs, which is a generalization of the stability theorem of Kodaira-Spencer,

4 Holomorphic Poisson structures

X: compact Kähler manifold with Kähler form ω

$\beta \in H^0(X, \wedge^2\Theta)$: holomorphic 2-vector filed

β: holomorphic Poisson structure

$\Leftrightarrow [\beta, \beta]_{\mathrm{Scou}} = 0$ (Scouten bracket)

A holomorphic Poisson str. β gives deformations of G.complex strs by $\mathcal{J}_{\beta t} := \mathrm{Ad}_{e^{\beta t}} \mathcal{J}_J$.

From theorem 3.3 (the stability theorem),

Theorem 4.1 A holomorphic Poisson structure β gives rise to non-trivial deformations of G.Kähler structures $(\mathcal{J}_{\beta t}, \psi_t)$.

rank β_x: the rank of 2-vector β_x, $x \in X$

$$\mathrm{Type}\,(\mathcal{J}_\beta)_x = n - 2\,\mathrm{rank}\,\beta_x$$

(X, J, ω) : compact Kähler manifold

$\Theta = T^{1,0}$

Deformations of generalized complex structures are parametrised by

$$H^2(X, \mathcal{O}_X) \oplus H^1(X, \Theta) \oplus H^0(X, \wedge^2\Theta)$$

Obstruction space to deformations of generalized complex structures

$$H^3(X, \mathcal{O}_X) \oplus H^2(X, \Theta) \oplus H^1(X, \wedge^2\Theta) \oplus H^0(\wedge^3\Theta)$$

If the obstruction space vanishes, then we obtain deformations of generalized Kähler structures parametrised by

$$H^2(X, \mathcal{O}_X) \oplus H^1(X, \Theta) \oplus H^0(X, \wedge^2\Theta) \oplus H^{1,1}(X)$$

In general there are obstructions to deformations to generalized complexstructures