Introduction to Dirac and generalized complex structures

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Symplectic and Poisson structures

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A *closed*, non-degenerate 2-form $\omega : TM \to T^*M$.
Induces a Poisson bracket by

$$\{f, g\} = \omega(df^{\sharp_\omega}, dg^{\sharp_\omega})$$

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How to unite them?

Real Dirac structures

Graphs of both ω and η are subbundles $L \subset TM \oplus T^*M$, isotropic (by skew-symmetry) w.r.t. the inner product

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Hence: need of an integrability condition.

B-transforms

The orthogonal group $\mathrm{SO}(TM \oplus T^*M, \langle,\rangle)$ has Lie algebra

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It acts by:

$$e^B(X + \alpha) = X + \alpha + \iota_X B.$$

The Courant bracket

Integrability

An almost Dirac structure L is a Dirac structure if it is closed under the Courant bracket:

$$[X+\alpha, Y+\beta]_C = [X, Y] + \mathcal{L}_X\,\beta - \mathcal{L}_Y\,\alpha - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha)$$

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- Does not satisfy the Jacobi identity.
- Reduces to usual bracket on vector fields.
- Vanishes on forms.

The Courant bracket as a derived bracket

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Thus

$$[X + \alpha, Y + \beta]_C = \mathrm{sk}([[X + \alpha, d], Y + \beta])$$

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Remark $\iota_{[X,X]} = [[\iota_X, d], \iota_Y]$, hence the usual bracket is itself a derived bracket.

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Similarly, if $F :=^* \pi(L)$, there exists a unique bivector $\eta \in \Lambda^2(TM)$ s.t.

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$e^B(L(E, \varepsilon)) = L(E, \varepsilon + i^*B), \quad i : E \hookrightarrow TM$
Locally, there exist B s.t. $e^B(L) = E \oplus \mathrm{Ann}E$.
Hence: Dirac geometry generalises foliations ($\varepsilon = 0$).

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From $L_{g\cdot\xi} = \rho(g)L_\xi$, $g \in \mathrm{Spin}(TM \oplus T^*M)$, follows that:
$\xi = e^B \wedge 1 = e^B$, $e^B\theta$ are again pure.

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$(X + \alpha) \cdot \xi = 0 \Leftrightarrow X \in E, \alpha \in \mathrm{Ann}(E)$.
Hence: $L(E, 0)$ is associated to the pure spinor line $\langle \det(\mathrm{Ann}(E)) \rangle \subset \Omega^k(M)$.

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Integrability in spinorial language (Gualtieri)

An almost Dirac structure L corresponding to the pure spinor line U is Courant involutive $\Leftrightarrow$ for any $\rho \in U$ there exists $X + \alpha \in (TM \oplus T^*M) \otimes \mathbb{C}$ such that

$$d\rho = (X + \alpha) \cdot \rho.$$

Push forward, pull back of (almost) Dirac structures

The linear setting

For a linear $\varphi : (V, L_V) \to (W, L_W)$, set:

$$\varphi_*(L_V) = \{\varphi(X) + \alpha \,|\, X + \varphi^*\alpha \in L_V\}$$
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(ii) If L_N is integrable and $\varphi^*(L_N) = L_M$, then L_M is integrable.

Local structure (cf. Courant, Bursztyn & Radko, Bursztyn & Weinstein)

Let L be a Dirac structure on M such that ${}^*\pi(L)$ is a subbundle of T^*M.

Then, locally, there exist submersions φ on M such that $\varphi_*(L)$ is a Poisson structure and $L = \varphi^*(\varphi_*(L))$.

These submersions are (germ) unique, up to Poisson diffeomorphisms of their codomains.

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Then $\eta(df, dg) = X(g) = -Y(f)$ and
$[X + df, Y + dg]_C = [X, Y] + d(\eta(df, dg))$, hence $\eta(df, dg)$ is basic w.r.t. $TM \cap L$.

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In terms of maximally isotropic subbundle

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Existence of $\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$ with
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In terms of $L(E, \varepsilon)$

Now $E = \pi(L) \subset TM \otimes \mathbb{C}$ and ε is complex.
$L(E, \varepsilon)$ is maximally isotropic. It has 0 real index $\Leftrightarrow$

- $E + \overline{E} = TM \otimes \mathbb{C}$ and
- $\mathrm{Im}(\varepsilon|_{E \cap \overline{E}})$ is non-degenerate.

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Every maximally isotropic $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ corresponds to a pure spinor line generated by a $\xi_L = e^{B+\mathrm{i}\omega}\Omega$.

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Examples

The extreme cases

Symplectic: $\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$.

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Nilmanifolds (Cavalcanti & Gualtieri)

5 classes of 6-dimensional nilmanifolds with no complex or symplectic structures.

All of them admit generalized complex structures.

Local structure around regular points

Generalized Darboux theorem (Gualtieri)

Locally, up to diffeomorphisms and B-field transforms, a generalized complex manifold is equivalent to $(\mathbb{C}^k, J_0) \times (\mathbb{R}^{2n-2k}, \omega_0)$.

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L is now represented by $\varphi^*\rho = e^A\Omega$ with $A = \varphi^*B + \mathrm{i}\varphi^*\omega$.

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L is now represented by $\varphi^*\rho = e^A\Omega$ with $A = \varphi^*B + \mathrm{i}\varphi^*\omega$.
Modify A such that $\varphi^*\rho = e^{B'+\mathrm{i}\omega_0}\Omega$ with B' real closed 2-form.

Normal form

Theorem

Locally, up to B-field transforms, $\mathcal{J}$ can be put into the *normal form*:

$$\mathcal{J} = \begin{pmatrix} F & \eta \\ -\omega & -F^* \end{pmatrix},$$

where

- F is an integrable f-structure:
  - $F^3 + F = 0$ (corresponds to J_0 extended with 0 on the transverse).
  - Accordingly, $TM = T^{1,0}M \oplus T^{0,1}M \oplus T^0M$.
  - integrability means that $T^{1,0}M$ and $T^{0,1}M \oplus T^0M$ are closed under Lie bracket.
- $\omega|_{T^0M}$ is symplectic, $\omega|_{T^{1,0}M \oplus T^{0,1}M} = 0$.
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f-structures will appear canonically on generalized Kähler manifolds.

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No satisfactory definition up to now.

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