Holomorphic maps between generalized complex manifolds

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based on joint work with Radu Pantilie

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Linear CR and co-CR maps

$t : (V, C_V) \to (W, C_W)$ linear s.t. $t(C_V) \subseteq C_W$.
$t : (V, D_V) \to (W, D_W)$ linear s.t. $t(D_V) \subseteq D_W$.

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$F \in \mathrm{End}(V)$ with $F^3 + F = 0$. Accordingly:

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Call $L(E \cap \overline{E}, \mathrm{Im}(\varepsilon|_{E\cap\overline{E}}))$ the *associated linear Poisson structure*.

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Normal form in tensorial language

The corresponding $\mathcal{J}$ is written as $\mathcal{J} = \begin{pmatrix} F & \eta \\ -\omega & -F^* \end{pmatrix}$.

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The only freedom is in the choice of F and, in fact, only the splitting of $V^0 \oplus V^{1,0}$ is to be chosen.

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- Up to B-transforms, L_V and L_W are in canonical form and t is Poisson f-linear.
- Up to B-transforms, $t = t_1 \oplus t_2$, with t_1 (resp. t_2) a Poisson (resp. complex) map between symplectic (resp. complex) vector spaces.

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- Composition preserves holomorphicity.

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Locally, around regular points, there exist submersions $\varphi : M \to P$ s.t. $\varphi_*(L)$ is a Poisson structure and $L = \varphi^*(\varphi_*(L))$.

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Theorem

Let (M, L_M) and (N, L_N) be regular real analytic generalized complex manifolds and let $\varphi : M \to N$ be a real analytic map.

If φ is holomorphic then, locally, up to the complexification of a real analytic B-field tranformation, the complexification of φ descends to a complex analytic Poisson morphism between the canonical Poisson quotients.

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The notion comes from harmonic morphisms.

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$\omega|_\mathcal{V}$ is non-degenerate, $\mathrm{Ker}\omega = \mathcal{H} \Rightarrow F$, ω are compatible.
Let $L = L(E, \mathrm{i}\omega)$ (in normal form), with $E = \mathcal{V} \oplus \mathcal{H}^{1,0}$.

L is integrable $\Leftrightarrow$ J is integrable, $\mathcal{V}$ has minimal leaves and $A(X, Y) := [X, Y]^v$ is $(1, 1)$ w.r.t. F.
For $n = 2$, L integrable $\Leftrightarrow$ φ is a harmonic morphism.

A generalized complex structure in normal form on a (M, g) s.t.:
- the associated f-structure is g-skew,
- the induced Poisson structure has rank 2,
- $\|\omega\| = 1$,

Inverse construction

Let $\varphi : (M^{n+2}, g) \to (N^n, J)$ be a p.h.c. submersion.
Let $\mathcal{V} = \mathrm{Ker}\varphi_*$, $\mathcal{H} = \mathcal{V}^\perp$, ω volume form on $\mathcal{V}$.
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Generalized Kähler manifolds

(M, L_1, L_2) s.t. $\mathcal{J}_1\mathcal{J}_2 = \mathcal{J}_2\mathcal{J}_1$ and $\mathcal{J}_1\mathcal{J}_2 < 0$.

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L_1 and L_2 Courant-integrable $\Leftrightarrow$ $L^\pm$ integrable $\Leftrightarrow$ $J_\pm$ integrable and parallel w.r.t. $\nabla^\pm := \nabla^g \pm \frac{1}{2} g^{-1} h$, $h = db$.

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Holomorphic functions on (M, L_1) resp. (M, L_2) are bi-holomorphic functions on (M, J_+, J_-) resp. $(M, J_+, -J_-)$.

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Corollary

Let (M, L_1^M, L_2^M) and (N, L_1^N, L_2^N) with $\mathcal{H}_M^+$ and $\mathcal{H}_N^+$ integrable.

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Corollary

Let (M, L_1^M, L_2^M) and (N, L_1^N, L_2^N) with $\mathcal{H}_M^+$ and $\mathcal{H}_N^+$ integrable.
Then any holomorphic $\varphi : (M, L_1^M) \to (N, L_1^N)$ descends, locally, w.r.t. the above Riemannian submersions, to a holomorphic map between the Kähler quotients.

A generalization of a theorem of Apostolov-Gualtieri

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Theorem

Any generalized Kähler manifold with $\mathcal{V} = 0$ (*i.e.* $[J_+, J_-] = 0$) is, up to a unique B-field transformation, locally given by the second product of two Kähler manifolds.
In particular, $h = db = 0$.

Generalized Kähler manifolds with $\mathcal{H}^- = 0$.

Tamed symplectic manifolds

(M, ε, J) s.t. $\varepsilon(JX, X) > 0$, J and $\varepsilon^{-1}J^*\varepsilon$ integrable, $d\varepsilon = 0$.

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Let ε be non-degenerate on M and J almost complex structure. Let $J_+ = J$, $J_- = -\varepsilon^{-1}J^*\varepsilon$. Let g, b be the symmetric and skew-symmetric parts of εJ.

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Example (cf. Hitchin '06)

Let (M, g, I, J, K) be hyperkähler and $\varepsilon := -(\omega_I + \omega_J)$.

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Here, $L_1 = L(TM^{\mathbb{C}}, 2\omega_I - \mathrm{i}(\omega_J - \omega_K))$, $L_2 = L(TM^{\mathbb{C}}, -\mathrm{i}(\omega_J + \omega_K))$

Generalized Kähler manifolds with $\mathcal{H}^- = 0$. Local description

Corollary

A g.K. manifold with $\mathcal{H}^+$ integrable and $\mathcal{H}^- = 0$ is, up to a unique B-transform, locally a product $(M \times N, L_1^M \times L_1^N, L_2^M \times L_2^N)$ where (L_1^M, L_2^M) comes from a Kähler structure on M and (L_1^N, L_2^N) is a g.K. structure on N with $J_+ + J_-$ and $J_+ - J_-$ invertible.

Generalized Kähler manifolds with $\mathcal{H}^- = 0$.

Induced holomorphic Poisson structure (cf. Hitchin '06)

For a g.K. (M, L_1, L_2) coming from a tamed symplectic structure, let $\rho^\pm : TM^\mathbb{C} \to T_\pm^{1,0}M$. Then $\rho_*^\pm(L_2)$ is a holomorphic Poisson structure on $(M, J_\pm)$.

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The associated Poisson bivectors on $(M, J_\pm)$ are

$$\eta_- = -\eta_+ = \frac{1}{4}[J_+, J_-]g^{-1}.$$

The symplectic foliation associated to η_+ is precisely $\mathcal{V}$.

Holomorphic maps between generalized Kähler manifolds with $\mathcal{H}^- = 0$.

Induced holomorphic Poisson morphism

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If

- $\varphi : (M, L_2^M) \to (N, L_2^N)$ is holomorphic and,
- at least one of $\varphi : (M, J_+^M) \to (N, J_+^N)$ and $\varphi : (M, J_-^M) \to (N, J_-^N)$ is holomorphic,

then φ is a holomorphic Poisson morphism between the associated holomorphic Poisson structures.