Neutral Hypercomplex Structures in Dimension Four

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ALGEBRA OF SPLIT QUATERNIONS

Denote by $\mathbb{H}' = \{a + bi + cs + dt \in \mathbb{R}^4 | i^2 = -1, s^2 = t^2 = 1, is = -si = t\}$ the algebra of the split quaternions. They are associated with a natural scalar product of split signature (2,2). For $p, q \in \mathbb{H}'$, the product is defined by $|p|^2 =< p, p >= a^2 + b^2 - c^2 - d^2$. Then one has:

$$pq + qp = -2 < p, q >$$

which is the definition of the Clifford algebra $\mathcal{C}^{(1,1)}$ determined by this scalar product. Other algebraic relations: $\overline{pq} = \overline{q}\ \overline{p}$ and $|pq|^2 = |p|^2|q|^2$.

NEUTRAL HYPERCOMPLEX STRUCTURES

Based on the algebra $\mathbb{H}'$ one defines a neutral hypercomplex structure (also called complex product and hyper-paracomplex structure) on M^{4n} as a triple of endomorphisms I, S, T of TM

with $I^2 = -Id, S^2 = T^2 = Id, IS = T = -SI$ satisfying the integrability condition $N_I = N_S = N_T = 0$, where $N_A(X,Y) = A^2[AX,AY]+[X,Y]-A[AX,Y]-A[X,AY]$ is the Nijenhuis tensor associated with $A = I, S, T$.

Moreover there exists a unique torsion-free connection (called the Obata connection) ∇ such that $\nabla I = \nabla S = \nabla T = 0$. The structure is the "split analog" of hypercomplex structure. For a given neutral hypercomplex structure (I, S, T) one can consider the set $K_{(a,b,c)} = aI + bS + cT$. Then $K^2_{(a,b,c)} = (-a^2 + b^2 + c^2)Id$ and $\nabla K_{(a,b,c)} = 0$. So in particular:

i) If $a^2 - b^2 - c^2 = 1$, $K_{(a,b,c)}$ is a complex structure,

ii) If $a^2 - b^2 - c^2 = -1$, $K_{(a,b,c)}$ is a product structure (called also paracomplex), *iii)* If $a^2 - b^2 - c^2 = 0$, $Ker(K_{(a,b,c)}) = Im(K_{(a,b,c)})$ is involutive middle-dimensional distribution on M.

A pseudo-Riemannian metric g, such that I, S, T are skew-symmetric is called neutral hyperhermitian (or hyper-parahermitian). Such metric has split signature.

Lemma 1 *Every neutral (almost) hypercomplex 4-manifold M has a double or 4-fold covering which admits a neutral (almost) hyperhermitian metric*

Example: If S is paracomplex, a neutral metric for which $g(SX, SY) = -g(X, Y)$ is called parahermitian. Any neutral hyperhermitian metric is parahermitian with respect to any $K_{(a,b,c)}$ from ii). However not every paracomplex manifold admits parahermitian metric.

The reason is that a parahermitian metric determines a global isomorphism $T^{+} \to (T^{-})^{*}$, where $T = T^{+} \oplus T^{-}$ is the splitting of the tangent bundle into ± 1-eigenbundles of S. Then the product $T^2 \times S^2$ with $S|_{S^2} = +Id$, $S|_{T^2} = -Id$ is an easy example of paracomplex manifold without compatible parahermitian metric.

For the next part of the talk we will concentrate on four-dimensional case M^4. A neutral hyperhermitian metric in this case is anti-self-dual.

There are two additional equivalent characterizations of 4-dimensional neutral hypercomplex manifold M^4:

Theorem 1 *Let M be an oriented 4-dimensional smooth manifold.*

i) M admits a neutral hypercomplex structure if and only if M admits two complex structures *with the same orientation* I_1 *and* I_2*, such that* $I_1 I_2 + I_2 I_1 = 2pId$ *for a constant* p *with* $|p| > 1$*.*

ii) M *admits a neutral hyperhermitian structure iff* M *admits three 2-forms* $\Omega_1, \Omega_2, \Omega_3$ *such that* $-\Omega_1^2 = \Omega_2^2 = \Omega_3^2 = vol, \quad \Omega_i \wedge \Omega_j = 0$ *for* $i \neq j$ *and a 1-form* θ *such that* $d\Omega_i = \theta \wedge \Omega_i$*.*

Remark: In i) above, if $|p| < 1$, then I_1, I_2 determine a usual hypercomplex structure. It is valid in any dimension. For any neutral hyperhermitian metric the set $KerK_{(a,b,c)}$ always determines a null-space (α-surface if the orientation is determined by I).

As a corollary from the theorem we also obtain that the conformal class of a neutral hyperhermitian metric is fixed by the neutral hypercomplex structure in dimension four.

In *ii)* the forms Ω_i are the fundamental forms of I, S, T for some neutral hyperhermitian metric.

NEUTRAL HYPERKÄHLER OR HYPERSYMPLECTIC STRUCTURES

If $\theta = 0$ the structure is called *neutral hyperkähler* or *hypersymplectic*. Then there is a neutral signature metric such that the fundamental forms of I, S, T are closed and its Levi-Civita connection coincides with the Obata connection. Such structures on compact complex surfaces with complex structure I are classified by H.Kamada - they can only be 4-tori or Kodaira surfaces. J.Petean and H.Kamada constructed such structures on both the 4-tori and primary Kodaira surfaces. We can notice the following:

Theorem 2 *Let* (M, g) *be a compact pseudo-Riemannian 4-manifold of signature* $(2, 2)$ *which admits two independent parallel null vector fields. Then* M *admits a neutral hyperkähler structure*

COMPACT COMPLEX SURFACES WITH VANISHING FIRST CHERN CLASS

The complex structure I of a neutral hypercomplex structures has a vanishing first Chern class. So a compact complex surface admits such a structure only if $c_1(I) = 0$. The classification of such surfaces is known:

Theorem 3 *Let (M, I) be a compact complex surface with complex structure I and $c_1(I) = 0$. Then (M, I) is minimal and one of the following: a torus, a K3-surface, a primary Kodaira surface, a Hopf surface, an Inoue surface or a properly elliptic surface with odd b_1.*

Theorem 4 *The following compact complex surfaces admit a neutral hypercomplex structure: the 4-tori, the primary Kodaira surfaces, the Inoue surfaces of type $S^{\pm}$, the properly elliptic surfaces with odd b_1 and the quaternionic Hopf surfaces.*

If the 1-form θ is closed, then the neutral hypercomplex structure is locally conformal to neutral hyperkähler. All known examples of neutral hypercomplex structures on the surfaces of the above theorem are locally conformally neutral hyperkähler. In this case there is:

Theorem 5 *Let (M, I) be a compact complex surface admitting a locally conformally neutral hyperkäher structure (g, I, S, T). Then M is one of the following: a torus, a primary Kodaira surface, an Inoue surface of type $S_N^{\pm}$ or a Hopf surface. All such surfaces with the exception of the non-quaternionic Hopf surfaces admit such a structure.*

Remark: All non-Kähler compact complex surfaces with vanishing first Chern class admit transitive actions of a 4-dimensional Lie group.

The above mentioned neutral hypercomplex structures could be chosen to be invariant. Invariant structures are found by N.Blazic and S.Vukmirovich and also A.Andrada and S.Salamon.

Examples

(1) Primary Kodaira surfaces (H.Kamada, J.Petean)

Consider the affine transformations $\rho_i(z_1, z_2) = (z_1 + a_i, z_2 + \overline{a}_i z_1 + b_i)$ of $\mathbb{C}^2$, where a_i,b_i, $i = 1, 2, 3, 4$, are complex numbers such that $a_1 = a_2 = 0, Im(a_3\overline{a}_4) = b_1$. Then ρ_i generate a group G of affine transformations acting freely and properly discontinuously on $\mathbb{C}^2$. The quotient space $\mathbb{C}^2/G$ is called a primary Kodaira surface. H. Kamada showed that in the complex coordinates (z_1, z_2) of $\mathbb{C}^2$ any neutral hyperkähler structure on the primary Kodaira surface is given by the following symplectic forms:

$$\Omega_1 = Im(dz_1 \wedge d\overline{z}_2) + iRe(z_1) dz_1 \wedge d\overline{z}_1 + (i/2)\partial\overline{\partial}\phi,$$

$$\Omega_2 = Re(e^{i\eta} dz_1 \wedge d\overline{z}_2),\ \Omega_3 = Im(e^{i\eta} dz_1 \wedge d\overline{z}_2),$$

where η is a real constant and ϕ is a smooth function on M such that:

$$4i(Im(dz_1 \wedge d\overline{z}_2) + iRe(z_1)(dz_1 \wedge d\overline{z}_1)) \wedge \partial\overline{\partial}\phi = \partial\overline{\partial}\phi \wedge \partial\overline{\partial}\phi$$

If ϕ is a function of z_1 and $\overline{z_1}$, one obtains the solution due to J.Petean. It also provides an infinite-dimensional family of neutral hyperkähler structures - a sharp contrast with the positive case

(2) Quaternionic Hopf surfaces:

Consider $M = (\mathbb{H}' - \{0\})/\mathbb{Z}$, where the action is generated by $L_a : q \to aq$, where $a \in \mathbb{C}$

is a complex number with $|a| > 1$. If $q = z_1 + z_2 s$, then the action is $(z_1, z_2) \to (az_1, \overline{a}z_2)$. Then the following forms define conformally neutral hyperkähler structure on $\mathbb{H}' - \{0\}$ which descends to a locally conformal neutral hyperkähler structure on M:

$$\Omega_1 = i\frac{dz_1 \wedge d\overline{z}_1 - dz_2 \wedge d\overline{z}_2}{|z_1|^2 + |z_2|^2}$$

$$\Omega_2 + i\Omega_3 = \frac{dz_1 \wedge dz_2}{|z_1|^2 + |z_2|^2}$$

If a is real one can take $\Omega_1 = Re\frac{dz_1 \wedge d\overline{z}_2 + i\partial\overline{\partial}\phi}{|z_1|^2 + |z_2|^2}$ for a function ϕ depending on $(z_1, \overline{z_1})$.

(3) Inoue surfaces of type $S^{\pm}$

These surfaces are factors of $\mathbb{C} \times H$, where H is the upper half-plane. We use the fact that they appear as quotients of a solvable Lie group G

by a discrete subgroups and the complex structure is invariant. There is a transitive action of G on $\mathbb{C} \times H$ given by:

$$\begin{pmatrix} \epsilon & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \\ 1 \end{pmatrix}$$

where $(z, w) \in \mathbb{C} \times H$ and $\alpha > 0, a, b, c$ are real numbers, $\epsilon = \pm 1$.

The Lie algebra G and the complex structure are defined as follows:

$$[X_2, X_3] = -X_1, [X_4, X_2] = X_2, [X_4, X_3] = -X_3$$

$$IX_1 = X_2, IX_2 = -X_1, IX_3 = X_4 - qX_2$$

$$IX_4 = -X_3 - qX_1$$

Suppose that α_i is the dual basis of X_i of left invariant 1-forms. Then

$$d\alpha_1 = \alpha_2 \wedge \alpha_3, d\alpha_2 = \alpha_2 \wedge \alpha_4, d\alpha_3 = -\alpha_3 \wedge \alpha_4, d\alpha_4 = 0$$

and

$$\Omega_2 + i\Omega_3 = (\alpha_1 + i\alpha_2 + iq\alpha_4) \wedge (\alpha_3 + i\alpha_4)$$

We see that $\theta = \alpha_4$ and $\Omega_1 = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4$ provides an invariant neutral hypercomplex structure. Using the description of the forms α_i in terms of complex coordinates on $\mathbb{C} \times H$, we can deform Ω_1 to $\Omega_1 + \frac{i\partial\bar{\partial}\phi}{Im(w)}$ for arbitrary function ϕ depending on $Im(w)$ (the imaginary part of the H coordinate).

(4) Properly elliptic surfaces.

These are complex surfaces with universal cover $\widetilde{SL_2(\mathbf{R})} \times \mathbf{R}^1$ and are dual in a sense to Hopf surfaces. The Lie algebra $sl_2(\mathbf{R}) \oplus \mathbf{R}^1$ and the hypercomplex structure is defined as follows:

$$[X_2, X_3] = X_1, [X_1, X_2] = X_3, [X_1, X_3] = X_2$$

$$IX_1 = X_3, IX_2 = X_4, SX_1 = X_2, SX_3 = -X_4$$

GENERALIZED PSEUDO KÄHLER STRUCTURES AND NEUTRAL BIHERMITIAN SURFACES

From Theorem 1, i), one can see that it is natural to consider a 4-manifold with two complex structures I_1, I_2 for which $I_1 I_2 + I_2 I_1 = 2pId$ for a function p with $|p| > 1$. In this case the structures are compatible with a neutral metric, while for $|p| < 1$ they are compatible with a positive-definite one. We can relate such structure with the generalized complex structures.

Definition 1 *A (twisted) generalized pseudo-Kähler structure is a pair of commuting (twisted) generalized complex structures* $J_1, J_2 T \oplus T^* \to T \oplus T^*$*, such that* $G = -J_1 J_2$ *has* ± 1*-eigenspaces* $L^{\pm}$ *transversal to* TM *and the canonical inner product on* $T \oplus T^*$ *is nondegenerate on* $L^{\pm}$*.*

Using the same proof as in M.Gualtieri's thesis, we have

Theorem 6 *A generalized pseudo-Kähler structure on a manifold M is equivalent to a quadruple (I_1, I_2, g, b) where g is a pseudo-metric and I_1, I_2 are g-hermitian and $d^1\omega_1 = -d^2\omega_2 = db$, where ω_i is the Kähler form of I_i.*

The triple (I_1, I_2, g) is called a neutral bihermitian structure, a strong neutral bihermitian structure if $I_1 \neq \pm I_2$ at any point, and a positive neutral bihermitian structure if both I_1 and I_2 have positive orientation. In dimension four a pair of commuting generalized complex structures gives rise to a strong positive bihermitian structure when they are determined by spinors $\phi_1 = e^{-B+i\omega_1}$ and $\phi_2 = e^{B+\omega_2}$ with closed 2-forms B, ω_1, ω_2 satisfying:

$$B\omega_1 = B\omega_2 = \omega_1\omega_2 = \omega_1^2 + \omega_2^2 - 4B^2 = 0$$

$$\omega_1^2 = \lambda\omega_2^2, \lambda \neq 0$$

Here $\lambda > 0$ determines a regular bihermitian structure and $\lambda < 0$ determines a neutral bihermitian structure.

Example: One can deform a neutral hyperkähler structure to obtain a generalized pseudo-Kähler structure similarly to the positive-definite case. Fix two complex structures in the neutral hypercomplex family I_1 and I_2. Their commutator K, after normalization is a product structure and defines a closed 2-form ω_K which is a common real part of two (2,0)-forms with respect to I_1 and I_2 say $\omega_K + i\omega'$ and $\omega_K + i\omega''$. Then $B = \omega_K, \omega_1 = \omega' + \omega'', \omega_2 = \omega' - \omega''$ will provide a generalized pseudo-Kähler structure with $db = 0$. However if we use a one-parameter group of ω_K-Hamiltonian transformations H_t, we can consider $B = \omega_K$, $\omega_1 = \omega' + H_t^*\omega'', \omega_2 = \omega' - H_t^*\omega''$. Then this is a generalized pseudo-Kähler structure with $db \neq$

0 in general. Notice that $(I_1)_t = I_1$ is unchanged and $(I_2)_t = H^*_{-t}(I_2)H^*_t$. The case of primary Kodaira surface is worth noticing, since it doesn't admit any positive generalized Kähler structure.

NEUTRAL HYPERCOMPLEX REDUCTION AND INSTANTON MODULI SPACES

The reduction of neutral hypercomplex structures is similar to the hypercomplex reduction as developed by D. Joyce and is based on the reduction of hypersymplectic structures considered by N.Hitchin. Let G be a compact group of hypercomplex automorphisms of (M, I, S, T) with Lie algebra $\mathfrak{g}$ and denote the algebra of the induced hyper-holomorphic vector fields with the same $\mathfrak{g}$. Suppose that $\nu = (\nu_1, \nu_2, \nu_3)$: $M \longrightarrow \mathbf{R}^3 \otimes \mathfrak{g}^*$ is a G-equivariant map satisfying the following:

i) The Cauchy-Riemann condition $Id\nu_1 = -Sd\nu_2 = -Td\nu_3$, and

ii) The transversality condition $Id\nu_1(X) \neq 0$ for all $X \in \mathfrak{g}$.

Any map satisfying these conditions is called a G-moment map. Given a point $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ in $\mathbf{R}^3 \otimes \mathfrak{g}$, denote the level set $\nu^{-1}(\zeta)$ by P. If ζ_i are in the center of $\mathfrak{g}$ then the level set P is invariant. Then we have:

Theorem 7 *Let ν be a G-moment map for a group G which acts properly and freely on $P = \nu^{-1}(0)$. Suppose that on $\nu^{-1}(0)$ there is no non-zero solution to the equation $IX + SY + TZ = 0$ for $X, Y, Z \in \mathfrak{g}$. Then the quotient manifold $N = P/G$ is smooth and inherits a neutral hypercomplex structure.*

We first notice that in the moment map definition one can use any 3 complex structures

I_1, I_2, I_3 of the family $K_{(a,b,c)}$ instead of I, S, T. Then the Cauchy-Riemann condition is $I_1 d\nu_1 = I_2 d\nu_2 = I_3 d\nu_3$. Here the anti-commutators of I_1, I_2, I_3 satisfy Theorem 1 i). Then the reduction theorem is still valid.

Now consider a compact complex surface with neutral hypercomplex structure and a neutral hyperhermitian metric. Then we fix complex structures I_1, I_2, I_3 and their Kähler forms $\omega_1, \omega_2, \omega_3$, which define a basis for the self-dual forms Λ^+ at each point. Now a 2-form F is ASD if and only if $F \wedge \omega_i = 0$ for $i = 1, 2, 3$. In particular a connection A on a SU(k)-bundle is an instanton if its curvature F_A satisfies this condition. Then we have:

Corollary 1 *The "smooth part" of the moduli space of SU(k)-instantons on a compact neutral hypercomplex four manifold admits a neutral hypercomplex structure.*

Here the group G is the gauge group of the bundle. The moment maps are given by $\nu_i(A) = F_A \wedge \omega_i$. If $a \in \Omega^1(M, su(k))$ is any tangent vector at A generated by $Lie(G)$, then $d(\nu_i)_A(a) = d_A a \wedge \omega_i$. In this case the main identity is $\omega_i \wedge d_A^c a = d_A * a - d^c\omega \wedge a$ for any complex structure where $d_A^c = I^{-1} d_A I$. The Cauchy-Riemann condition follows from the identity $d^1\omega_1 = d^2\omega_2 = d^3\omega^3 = *\theta$ satisfied for any neutral hyperhermitian structure. Then the subset in the smooth part of the moduli space where this structure is degenerate is given by $[A]$ such that $d_A^1 a + d_A^2 b + d_A^3 c = 0$ has a nonzero solution (a, b, c) for some $A \in [A]$.