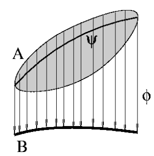
Home assignment 1: Zorn lemma

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solution with your monitor. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

Definition 1.1. Let $\phi: A \longrightarrow B$ be a surjective map of sets. **A section** of ϕ is a map $\psi: B \longrightarrow A$ such that $\psi \circ \phi = \mathsf{Id}$.



Definition 1.2. **Axiom of Choice** (AC) states that each surjective map admits a section.

Definition 1.3. **Partial order** is a relation $x \prec y$, which is **transitive** (if $x \prec y$ and $y \prec z$ then $x \prec z$) and **non-reflexive** ($z \prec z$ does not hold for any z). A set with partial order is called **partially ordered set**, or **poset**.

Definition 1.4. A partial order on S is called **total order** if for all $x \neq y$ either $x \prec y$ or $y \prec x$. A totally ordered poset S is called **well ordered** if any subset $S_1 \subset S$ has a minimal element (an element $v \in S_1$ such that $v \prec v'$ for all $v' \in S_1$ distinct from v).

Definition 1.5. Two posets A, B are called **isomorphic** if there exists a bijection from A to B preserving the order, with the inverse also preserving the order. Isomorphism classes of well ordered sets are called **ordinals**, or **ordinal numbers**. **Finite ordinals** are the same as natural numbers.

Remark 1.1. One can **add** the ordinals by taking a union $X \coprod Y$ of two well ordered sets and setting $X \prec Y$. Also, the ordinals can be multiplied: order on a product

$$(x, y) \prec (x', y') \text{ if } x \prec x', \text{ or } x = x', y \prec y'.$$

Exercise 1.1 (*). Prove that addition of ordinals is non-commutative.

Exercise 1.2. Prove that addition of ordinals is associative.

Exercise 1.3 (*). Prove that multiplication of ordinals is non-commutative.

Definition 1.6. **Interval** in a totally ordered set S is $[a, b] := \{x \in S \mid a \prec x \prec b\}$ **Initial interval** is an interval starting from the minimal element.

Exercise 1.4 (!). Let X, Y be well ordered sets. Prove that X is isomorphic a to an initial interval of Y, or Y is isomorphic a to an initial interval of X.

Exercise 1.5. Prove that such an isomorphism is unique.

Definition 1.7. **Zermelo theorem** states that any set admits well ordering.

Exercise 1.6. Deduce the Axiom of Choice from the Zermelo theorem.

Hint. The section ψ will map $b \in B$ to the minimal element in $\phi^{-1}(b)$.

Definition 1.8. Let $(S, \prec)$ be a poset. An element $x \in S$ is called **maximal** if there is no $y \in S$ such that $x \prec y$. For a subset $S_1 \subset S$ and $x \in S$, we write $S_1 \prec x$ if $\xi \prec x$ for all $\xi \in S_1$. **Zorn lemma** states that any poset $(S, \prec)$ has a maximal element if for any well ordered subset $S_1 \subset S$ there exists $x \in S_1$ such that $S_1 \backslash \{x\} \prec x$.

Exercise 1.7 (!). Deduce Zermelo theorem from Zorn lemma.

Hint. Let A be a set which we want to endow with a well ordering. Take for S the set of subsets of A equipped with well ordering, and write $S_1 \prec S_2$ if S_1 is an initial interval of S_2. Prove that the poset S satisfies assumptions of Zorn lemma, and any maximal element of S is A with well ordering.

Exercise 1.8 (!). Let A be a set, and S the set of all subsets $A_0 \subset A$ equipped with a well ordering $\prec_{A_0}$. Assume that S does not contain A. We intend to deduce Zermelo theorem from Axiom of Choice by absurd.

a. Using Axiom of Choice, construct a map $\phi : S \longrightarrow A$ mapping $W \in S$ to an element x in the complement $A \backslash S$.

b. Let $R \subset S$ be the set of all well ordered subsets $(W, \prec_W) \subset A$ such that for any initial interval $W_0 \subset W$, the minimal element of the complement $W \backslash W_0$ is $\phi(W_0)$. Prove that R is well ordered by inclusion.

c. Prove that the union of all $W \in R$ is well ordered.

Exercise 1.9 (!). Deduce Zorn lemma from Axiom of Choice, as follows. Like for Zermelo theorem, we prove Zorn lemma ad absurdum. Let $(A, \prec)$ be a poset without a maximal element, and S the set of well ordered subsets of A.

a. Prove that for any well ordered $W \subset A$, there exists $x \in A$ which satisfies $W \prec x$. Using Axiom of Choice, find a function $\phi : S \longrightarrow A$ such that $W \prec \phi(W)$ for all W.

b. Let $R_\phi \subset S$ be the set of all well ordered subsets $W \subset A$ such that for any initial interval $W_0 \subset W$, the minimal element of $W \backslash W_0$ is $\phi(W_0)$. Prove that R is well ordered by inclusion.

c. Prove that the union of all $W \in R$ is well ordered.