Home assignment 5: Hermitian structures

Rules: This is an assignment for the next week. Please solve all exercises and discuss your solutions with your monitor. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

5.1 Complex structures

Definition 5.1. Let V be a vector space over the field k. A linear operator $A \in \mathrm{Hom}(V,V)$ is called **semisimple** if it is diagonalizable over the algebraic closure of k.

Exercise 5.1 (!). Let V be a real vector space with positive definite scalar product. Prove that any orthogonal operator on V is semisimple.

Hint. Use the orthogonal decomposition.

Definition 5.2. Let A be an automorphism of a vector space V. A scalar product g on V is called **A-invariant** if $g(x,y) = g(Ax, Ay)$ for any $x, y \in V$.

Exercise 5.2. Let V be a vector space over $\mathbb{R}$, and $A \in \mathrm{Hom}(V,V)$ a linear operator of finite order, $A^n = \mathsf{Id}$.

a. Prove that V admits an A-invariant scalar product.

b. (!) Prove that A is semisimple

Definition 5.3. Let V be a vector space over $\mathbb{R}$. **A complex structure operator** is a linear map $I: V \longrightarrow V$ which satisfies $I^2 = -\mathsf{Id}_V$.

Exercise 5.3. Let $I: V \longrightarrow V$ be a complex structure operator on a vector space $V = \mathbb{R}^n$.

a. Prove that I is semisimple.

b. Prove that n is even, $n = 2m$, and I has m eigenvalues $\sqrt{-1}$ and m eigenvalues $-\sqrt{-1}$.

Exercise 5.4. Let (V, I) be a real vector space with a complex structure operator, $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ the eigenspace associated with the eigenvalue $\sqrt{-1}$, and $V^{0,1}$ the eigenspace of $-\sqrt{-1}$. Prove that $V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$, and the projection of $V \otimes_{\mathbb{R}} \mathbb{C}$ to $V^{1,0}$ along $V^{0,1}$ induces an isomorphism from $V \subset V \otimes_{\mathbb{R}} \mathbb{C}$ to $V^{1,0}$.

Exercise 5.5 (!). Let $\mathcal{C}$ be the category of real vector spaces equipped with a complex structure I, with morphisms given by linear maps compatible with I. Construct an equivalence of categories between $\mathcal{C}$ and the category of complex vector spaces.

Hint. To construct the functor from $\mathcal{C}$ to complex vector spaces, use the previous exercise. To construct a functor from complex vector spaces to $\mathcal{C}$, take a complex vector space, consider it as a real vector space, and put $I(v) = \sqrt{-1}v$.

Exercise 5.6 (!). Let V be an even-dimensional real space. Consider the complex conjugation on $V \otimes_{\mathbb{R}} \mathbb{C}$ acting in a usual way, that is, $\overline{\sum_i \lambda_i v_i} = \sum \bar{\lambda}_i v_i$, where $\lambda_i \in \mathbb{C}$ and $v_i \in V$. Let $W \subset V \otimes_{\mathbb{R}} \mathbb{C}$ be a subspace such that $W \oplus \overline{W} = V \otimes_{\mathbb{R}} \mathbb{C}$, where $\overline{W}$ denotes the complex conjugated space. Prove that there exists a complex structure operator $I \in \mathrm{End}(V)$ such that $W = V^{1,0}$. Prove that such I is uniquely determined by W.

5.2 Hermitian structures

Definition 5.4. Let (V, I) be a real vector space with a complex structure operator. An I-invariant positive definite scalar product g on (V, I) is called **an Hermitian metric**, or **Hermitian structure** and (V, I, g) – **an Hermitian space**.

Exercise 5.7. Prove that any (V, I) admits a Hermitian metric.

Exercise 5.8. Let (V, I, g) be a Hermitian space.

a. Prove that the form $\omega(x, y) := g(x, Iy)$ is anti-symmetric, that is, satisfies $\omega(x, y) = -\omega(y, x)$.

b. Prove that $\omega(x, Iy) = -\omega(Ix, y)$.

c. (!) Let (V, I) be a vector space equipped with a complex structure operator, and ω an antisymmetric form, satisfying $\omega(x, Iy) = -\omega(Ix, y)$. Suppose that $\omega(Ix, x) > 0$ for any non-zero $x \in V$. Prove that $g(x, y) := \omega(Ix, y)$ is a Hermitian form on (V, I).

Exercise 5.9 (!). Let (V, I, g) be a Hermitian space. Prove that every two elements of the triple g, I, ω determinee the third element uniquely.

Exercise 5.10. Let (V, I, g) be a Hermitian space, and $h(x, y) := g(x, y) + \sqrt{-1}\omega(x, y)$. Using the equivalence defined above, we consider (V, I) as a complex vector space. Prove that $h(x, x)$ is a real number, positive for each non-zero $x \in V$, and $h(\lambda x, y) = h(x, \bar\lambda y) = \lambda h(x, y)$ for any $\lambda \in \mathbb{C}, x, y \in V$.

Exercise 5.11 (!). Let V be a complex vector space, and $h : V \times V \longrightarrow \mathbb{C}$ a bilinear over $\mathbb{R}$ form which satisfies $h(x, y) = \overline{h(y, x)}$, and $h(\lambda x, y) = h(x, \bar\lambda y) = \lambda h(x, y)$ for any $\lambda \in \mathbb{C}, x, y \in V$. Assume that $h(x, x)$ is a real number, positive for each non-zero $x \in V$. Prove that $h(x, y) := g(x, y) + \sqrt{-1}\omega(x, y)$, for some Hermitian metric g on (V, I), where $I(x) = \sqrt{-1}x$.

Definition 5.5. Let (V, I, g) be a Hermitian space, and $h(x, y) := g(x, y) + \sqrt{-1}\omega(x, y)$. **An orthonormal basis** is a basis z_i in V such that $h(z_i, z_j) = \delta_{ij}$, that is $h(z_i, z_i) = 1$ and $h(z_i, z_j) = 0$ for $i \neq j$.

Exercise 5.12 (!). Let (V, I, g) be a Hermitian space, $\dim_{\mathbb{R}} V = 2n$. Prove that an orthonormal basis $z_1, ..., z_n$ always exists. Prove that it is a basis in V considered as a complex vector space, with $\sqrt{-1}x = I(x)$.

Exercise 5.13 (*). Let $V = \mathbb{R}^4$ be a 4-dimensional vector space equipped with a scalar product g. Construct a natural bijection between $S^2 \coprod S^2$ and the space of all orthogonal complex structures on V.

Definition 5.6. Let $g_{\mathbb{C}}$ be a non-degenerate bilinear symmetric form $2n$-dimensional complex vector space W. A complex subspace $W_1 \subset W$ is called **isotropic** if the restriction $g_{\mathbb{C}}\Big|_{W_1}$ vanishes.

Exercise 5.14. Let V be a real vector space equipped with a scalar product g, and $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$.

a. Prove that there exists a unique complex-linear bilinear symmetric form $g_{\mathbb{C}}$ on $V_{\mathbb{C}}$ such that $g_{\mathbb{C}}\Big|_V = g$.

b. (*) Let I be a complex structure operator on V and $V^{1,0} \subset V_{\mathbb{C}}$ the eigenspace of eigenvalue $\sqrt{-1}$. Prove that $V^{1,0}$ is isotropic with respect to $g_{\mathbb{C}}$.

c. (*) Construct a bijection between $g_{\mathbb{C}}$-isotropic subspaces $W \subset V_{\mathbb{C}}$, $\dim_{\mathbb{C}} W = \frac{1}{2}\dim_{\mathbb{C}} V_{\mathbb{C}}$, and g-orthogonal complex structure operators on V.