Home assignment 6: Tensor product

Rules: This is an assignment for the next week. Please solve all exercises and discuss your solutions with your monitor. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

6.1 Tensor product of R-modules

Remark 6.1. All rings are assumed to be commutative and with unity.

Definition 6.1. Let R be a ring, and M, M' modules over R. We denote by $M \otimes_R M'$ an R-module generated by symbols $m \otimes m'$, $m \in M, m' \in M'$, modulo relations $r(m \otimes m') = (rm) \otimes m' = m \otimes (rm')$, $(m + m_1) \otimes m' = m \otimes m' + m_1 \otimes m'$, $m \otimes (m' + m'_1) = m \otimes m' + m \otimes m'_1$ for all $r \in R, m, m_1 \in M, m', m'_1 \in M'$. Such an R-module is called **the tensor product of M and M' over R.**

Definition 6.2. Let M_1, M_2, M be modules over a ring R. **Bilinear map** $\mu(M_1, M_2) \xrightarrow{\phi} M$ is a map satisfying $\phi(rm, m') = \phi(m, rm') = r\phi(m, m')$, $\phi(m{+}m_1, m') = \phi(m, m'){+}\phi(m_1, m')$, $\phi(m, m'{+}m'_1) = \phi(m, m'){+}\phi(m, m'_1)$.

Exercise 6.1. (universal property of tensor product)
Construct a natural bijective correspondence between the set of homomorphisms $\operatorname{Hom}_R(M_1 \otimes_R M_2, M)$ and the set of bilinear maps $\operatorname{Bil}(M_1 \times M_2, M)$.

Exercise 6.2. Find non-zero R-modules A, B, such that $A \otimes_R B = 0$ for

a. $R = \mathbb{C}[t]$.

b. R is the ring of complex analytic functions on $\mathbb{C}$.

c. $R = \mathbb{Z}$.

d. $R = \mathbb{Z}/10$.

Definition 6.3. Let M, M' be R-modules. Consider the group $\operatorname{Hom}_R(M, M')$ of R-module homomorphisms. We consider $\operatorname{Hom}_R(M, M')$ as an R-module, using $r\phi(m) := \phi(rm)$. This R-module is called **internal** Hom **functor**, denoted $\mathcal{H}om_R$.

Exercise 6.3. Prove that $\mathcal{H}om_R(M_1 \otimes_R M_2, M) = \mathcal{H}om_R(M_1, \mathcal{H}om_R(M_2, M))$.

Exercise 6.4. Let $B : M_1 \times M_2 \longrightarrow M$ be a bilinear map of R-modules.

a. Prove that there exists a unique homomorphism $b : M_1 \otimes M_2 \longrightarrow M$, making the following diagram commutative:

$$\begin{tikzcd}
M_1 \times M_2 \arrow[r, "B"] \arrow[dr, "\tau"] & M_1 \otimes M_2 \arrow[d, "b"] \\
& M
\end{tikzcd}$$

b. Prove that this property defines $M_1 \otimes M_2$ uniquely.

6.2 Exact sequences and $\mathcal{H}om$-functor

Definition 6.4. **Exact sequence** of R-modules is a sequence of homomorphisms

$$... \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+2} \longrightarrow ...$$

(finite or infinite) such that the kernel of n-th arrow is the image of $n-1$-th arrow for all n. **Short exact sequence** is an exact sequence of form $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$.

Exercise 6.5. Let $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. Prove that the sequence

$$0 \longrightarrow \mathcal{H}om_R(M_3, N) \longrightarrow \mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N)$$

is exact, for any R-module N.

Exercise 6.6. Let $R = \mathbb{Z}$. Find an example of an injective homomorhism $M_1 \longrightarrow M_2$ such that $\mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N)$ is not surjective for some R-module N.

Definition 6.5. **An exact functor** F is a functor on the category of R-modules mapping any exact sequence of R-modules to an exact sequence.

Exercise 6.7 (*). Let F be a functor which maps any short exact sequence to an exact sequence. Prove that F is exact.

Definition 6.6. An R-module is called **injective** if the functor

$$M \longrightarrow \mathcal{H}om_R(M, N)$$

is exact.

Exercise 6.8. Find a non-zero injective module over $\mathbb{Z}$.

Exercise 6.9 (*). Find a non-zero injective module over $\mathbb{Z}$ containing $\mathbb{Z}/n\mathbb{Z}$.

Exercise 6.10. Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$ be an exact sequence of R-modules. Prove that the corresponding sequence

$$0 \longrightarrow \mathcal{H}om_R(N, M_1) \longrightarrow \mathcal{H}om_R(N, M_2) \longrightarrow \mathcal{H}om_R(N, M_3)$$

is exact for any R-module N.

Exercise 6.11. Let $R = \mathbb{Z}$. Find an epimorphism (surjective homomorphism) of R-modules $M_2 \longrightarrow M_3$ such that the corresponding map

$$\mathcal{H}om_R(N, M_2) \longrightarrow \mathcal{H}om_R(N, M_3)$$

is not surjective for some R-module N.

Definition 6.7. An R-module M is called **projective** if the functor $M \longrightarrow \mathcal{H}om_R(N, M)$ is exact.

Exercise 6.12 (!). Prove that a finitely generated R-module N is projective if and only if it is a direct summand of a free module R^n, that is, there is a direct sum decomposition of R-modules $R^n = N \oplus N'$.

Hint. Consider the surjective homomorphism

$$\mathcal{H}om_R(N, M_2) \longrightarrow \mathcal{H}om_R(N, M_3)$$

where $M_3 = N$, and M_2 is a free module equipped with a surjective homomorphism to N.

Exercise 6.13 (*). Suppose that R is a finitely generated ring over $\mathbb{C}$, and any finitely generated R-module is projective. Prove that R is a direct sum of fields, or find a counterexample.

6.3 Tensor product and exact sequences

Exercise 6.14. Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. Prove that for any R-modules N, N', the sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_3, N)) \longrightarrow \\ \longrightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_2, N)) \longrightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_1 N)) \end{aligned}$$

is exact

Exercise 6.15. Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. Prove that for any R-modules N, N', the sequence

$$0 \longrightarrow \mathcal{H}om_R(M_3 \otimes N', N) \longrightarrow \mathcal{H}om_R(M_2 \otimes N', N) \longrightarrow \mathcal{H}om_R(M_1 \otimes N', N)$$

is exact.

Definition 6.8. **A complex** of R-modules is a sequence $M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \xrightarrow{d_3} \ldots$ such that $d_i \circ d_{i+1} = 0$. **Cohomology** of this complex are the quotient groups, $H^i(M_*) := \frac{\ker(d_i)}{\operatorname{im} d_{i-1}}$. A complex is **exact** if it has zero cohomology.

Exercise 6.16. Consider a complex $M_1 \xrightarrow{\mu} M_2 \xrightarrow{\rho} M_3 \longrightarrow 0$ of R-modules such that the corresponding sequence

$$0 \longrightarrow \mathcal{H}om_R(M_3, N) \xrightarrow{\rho_N} \mathcal{H}om_R(M_2, N) \xrightarrow{\mu_N} \mathcal{H}om_R(M_1, N) \qquad (6.1)$$

is exact for any R-module N. Prove that E is also exact

Hint. Use injectivity of ρ_N to prove surjectivity of ρ by setting $N := M_3/\operatorname{im}\rho$. To prove exactness of E in the second term, use $N = M_2/\operatorname{im}\mu$ and apply exactness of the sequence (6.1) in the second term.

Exercise 6.17 (*). Let $E = \left(... \longrightarrow M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \longrightarrow ...\right)$ be a complex of R-modules such that

$$... \longrightarrow \mathcal{H}om_R(M_3, N) \longrightarrow \mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N) \longrightarrow ...$$

is exact for all R-modules N. Prove that E is also exact.

Exercise 6.18 (!). Let $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. Prove that the sequence

$$M_1 \otimes_R M \longrightarrow M_2 \otimes_R M \longrightarrow M_3 \otimes_R M \longrightarrow 0$$

is also exact.

Hint. Use Exercise 6.16, and apply exactness of

$$0 \longrightarrow \mathcal{H}om_R(M_3 \otimes M, N) \longrightarrow \mathcal{H}om_R(M_2 \otimes M, N) \longrightarrow \mathcal{H}om_R(M_1 \otimes M, N),$$

using an isomorphism $\mathcal{H}om_R(M_1 \otimes_R M_2, M) = \mathcal{H}om_R(M_1, \mathcal{H}om_R(M_2, M))$ (Exercise 6.3).

Exercise 6.19 (!). Let $R = \mathbb{C}[t]$. Find an exact sequence of R-modules $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ and an R-module N such that the corresponding map $M_1 \otimes_R M \longrightarrow M_2 \otimes_R M$ is not injective.

Exercise 6.20 (!). Let $I \subset R$ be an ideal. Prove that for any R-module M, one has $M \otimes_R (R/I) \cong M/IM$.

Hint. Take an exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ and apply the functor $\otimes M$.

Exercise 6.21 (!). Let I, I' be distinct maximal ideals in a ring R. Prove that $R/I \otimes_R R/I' = 0$.

Exercise 6.22 (!). Let $R^n \longrightarrow R^m$ be a surjective homomorphism of R-modules. Prove that $n \geqslant m$.