Home assignment 8: Group representations

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solution with your monitor. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

8.1 Group representations

Definition 8.1. **Representation of a group** G, or G**-representation** is a group homomorphism $G \longrightarrow GL(V)$. In this case, V is called **representation space**, and **representation**. One also says that **the group** G **acts on the vector space** V. **Morphism** of G-representations is a linear map which is compatible with the action of G.

Definition 8.2. **Irreducible representation** is a representation having no proper G-invariant subspaces. **Semisimple representation** is a direct sum of irreducible ones.

Exercise 8.1. Let G be a group acting on vector spaces V, V'. Define the action of G on $V \otimes V'$ via $g(v \otimes v') = g(v) \otimes g(v')$. Prove that this defines an action of G on $V \otimes V'$.

Remark 8.1. In this case, we say that the action of G is extended to $V \otimes V'$ **by multiplicativity** or **multiplicatively**.

Definition 8.3. Let V be a representation of G. A vector $v \in V$ is called G**-invariant** if $g(v) = v$ for any $g \in V$. The space of all G-invariant vectors is denoted V^G. A scalar product h on V is called G**-invariant** if h is G-invariant as a vector in $V^* \otimes V^*$.

Exercise 8.2. Let V be a representation of a finite group G over a field $k \subset \mathbb{R}$.

a. Prove that V admits a G-invariant positive definite scalar product.

b. Prove that any finite-dimensional representation of G over $\mathbb{R}$ and $\mathbb{C}$ is semisimple.

Exercise 8.3. Find a d-dimensional G-representation V over $\mathbb{R}$ and positive numbers n, m such that $n + m = d$, and V does not admit a G-invariant bilinear symmetric form of signature (n, m).

Exercise 8.4. Let $\nu \in \operatorname{End}_G(V)$ be a non-zero endomorphism of a finite-dimensional irreducible G-representation V. Prove that V is an isomorphism.

Exercise 8.5. Let V be a finite-dimensional irreducible group representation, and $\operatorname{End}_G(V)$ its automorphism algebra. Prove that $\operatorname{End}_G(V)$ is a division algebra (all its non-zero elements are invertible).

Remark 8.2. *Frobenius theorem* claims that all division algebras over $\mathbb{R}$ are isomorphic to complex numbers, real numbers and quaternions.

Definition 8.4. Let V be a finite-dimensional irreducible group representation over $\mathbb{R}$. It is called **real** if $\operatorname{End}_G(V) = \mathbb{R}$, **complex** if $\operatorname{End}_G(V) = \mathbb{C}$, and **quaternionic** if $\operatorname{End}_G(V)$ is the algebra of quaternions.

Exercise 8.6 (!). Find examples of irreducible finite group representations which are real, complex, and quaternionic.

Exercise 8.7 (*). Let V be a finite-dimensional representation of a finite group. Prove that V does not admit an invariant non-degenerate bilinear symmetric form which is not sign-definite[1].

[1] Sign-definite means positive or negative definite.

Exercise 8.8. Consider the action of symmetric group Σ_n on $V = R^n$, given by permutation of coordinates.

a. Find the space of G-invariants V^G.

b. (*) Extend the G-action ot $V \otimes V$ by multiplicativity. Find the space of G-invariants $(V \times V)^G$.

8.2 Group algebra

Definition 8.5. Let G be a group, k a field. **Group algebra** of G over k is a vector space $k[G]$ freely generated by elements of G (that is, consisting of the linear combinations $\sum \lambda_i g_i$ ($\lambda_i \in k$, $g_i \in G$) and with multiplication given by the formula

$$(\sum \lambda_i g_i)(\sum \lambda'_j g'_j) = \sum_{i,j} \lambda_i \lambda'_j g_i g'_j.$$

One may consider $k[G]$ as a G-representation in three different ways: **left action** $g(\sum \lambda_i g_i) = \sum \lambda_i g g_i$, **right action** $g(\sum \lambda_i g_i) = \sum \lambda_i g_i g^{-1}$, and **adjoint action** $g(\sum \lambda_i g_i) = \sum \lambda_i g g_i g^{-1}$

Exercise 8.9. Let G be a finite group, and $V = k[G]$ with the left G-action. Prove that $\dim_k \operatorname{End}_G(V) = |G|$.

Exercise 8.10. a. Consider $k[G]$ as a G-representation with the left action of G. Prove that for any irreducible G-representation V there exists a surjective G-morphism $\phi \in \operatorname{Hom}_G(k[G], V)$

b. (!) Prove that $\dim \operatorname{Hom}_G(k[G], V) = \dim V$.

c. (*) Assume, in addition, that $k = \mathbb{C}$, and G is finite. Prove that $k[G]$, as a G-representation with the left action, is isomorphic to direct sum of all irreducible representations V_i of G, with each of V_i taken $d = \dim_k V_i$ times:

$$k[G] \cong \bigoplus V_i^{\dim V_i}.$$

Exercise 8.11. Consider the group algebra Σ_3 of the symmetric group. Find all irreducible representations of Σ_3.

8.3 Representations of commutative groups

Definition 8.6. A flag on a d-dimensional vector space V is a family of subspaces $0 = V_0 \subset V_1 \subset ... \subset V_d = V$ with $\dim V_i = i$.

Exercise 8.12. Let $A, B \in \operatorname{End} V$ be commuting operators on a complex vector space V. Prove that they have a common eigenvector.

Exercise 8.13. Let $A_1, ..., A_n$ be a family of commuting operators on a complex vector space V. Prove that there exists a flag which is preserved by all A_i.

Hint. Use the previous exercise.

Exercise 8.14. Let V be a complex representation of a commutative finite group G. Prove that V is a direct sum of 1-dimensional representations.

Exercise 8.15. Let $G = \mathbb{Z}/n\mathbb{Z}$ be a finite cyclic group.

a. (!) Prove that $\mathbb{C}[G] = \mathbb{C}^n$ (direct sum of n copies of $\mathbb{C}$).

b. (*) Prove that $\mathbb{R}[G] = \mathbb{C}^d \oplus \mathbb{R}$ if $n = 2d + 1$ and $\mathbb{R}[G] = \mathbb{C}^d \oplus \mathbb{R}^2$ if $n = 2d + 2$.