Class assignment 9: Exercises in divisibility (2)

Rules: This is a test to be done in class during your monitor session. Please solve it in class and give to your monitor Renan Santos. Your solutions will have little influence on the final grade. If you use some result, please state it and (unless it's trivial) explain the proof.

Definition 9.1. An element p of a ring R is called **prime** if for any decomposition $p = p_1 p_2$, either p_1 or p_2 is invertible. We say that a ring R **admits the prime decomposition** if any element is a product of primes. R is called **factorial** if any element $r \in R$ admits the prime decomposition $r = \prod_i p_i^{\alpha_i}$, and this decomposition is unique up to invertible factors and permutation of p_i.

Exercise 9.1. Consider the ring $R := \mathbb{Z}[\sqrt{-3}]$. Prove that R is not factorial.

Exercise 9.2. Construct a ring A and a prime element $p \in A$ such that the principal ideal (p) is not prime.

Exercise 9.3. Consider a ring $R \subset \mathbb{C}$, which is discrete as a subset of $\mathbb{C}$. Prove that R admits the prime decomposition.

Exercise 9.4. Let $R := \mathbb{Z}[\sqrt{p}]$, where $p \in \mathbb{Z}$ is a prime number. Prove that R has no zero divisors.

Exercise 9.5. Consider the ring $R := \mathbb{Z}[\sqrt{p}]$, where $p \in \mathbb{Z}$ is a prime number. Prove that R admits the prime decomposition.

Exercise 9.6. Find whether the following numbers $p \in \mathbb{Z}$ are prime in the ring $\mathbb{Z}[\sqrt{-5}]$. Give the proof.

a. $p = 7$.

b. $p = 13$.

c. $p = 41$.

Exercise 9.7. Find a non-principal ideal in the ring $\mathbb{Z}[\sqrt{-p}]$, for some prime $p > 5$.