Class assignment 11: Group representations

Rules: This is a test to be done in class during your monitor session. Please solve it in class and give to your monitor Renan Santos. Your solutions will have little influence on the final grade. If you use some result, please state it and (unless it's trivial) explain the proof.

Definition 11.1. Representation of a group G, or G**-representation** is a group homomorphism $G \longrightarrow GL(V)$. **Irreducible representation** is a representation having no proper G-invariant subspaces. **Semisimple representation** is a direct sum of irreducible ones.

Exercise 11.1. Consider the group G of non-degenerate upper triangular matrices $n \times n$. Construct a non-semisimple representation of G.

Exercise 11.2. Construct a finite group G and a non-semisimple representation $G \longrightarrow GL(V)$, where V is a finite-dimensional vector space over a finite field.

Definition 11.2. Let V be a finite-dimensional irreducible group representation over $\mathbb{R}$. It is called **real** if $\operatorname{End}_G(V) = \mathbb{R}$, **complex** if $\operatorname{End}_G(V) = \mathbb{C}$, and **quaternionic** if $\operatorname{End}_G(V)$ is the algebra of quaternions.

Exercise 11.3. Find an example of a quaternionic representation of a finite group.

Exercise 11.4. Find a quaternionic representation of a cyclic group of order 17, or prove that such a representation cannot exist.

Exercise 11.5 (2 points). Find a quaternionic representation of the group G of isometries of a regular tetrahedron, or prove that such a representation cannot exist.

Exercise 11.6. Let $V = \mathbb{Q}^2$ be an irreducible 2-dimensional representation of the cyclic group $G = \mathbb{Z}/n$ over $\mathbb{Q}$.

a. Prove that the algebra $\operatorname{End}_G(V)$ of endomorphisms of this representation is commutative and 2-dimensional.

b. (2 points) Prove that $\operatorname{End}_G(V)$ is a number field isomorphic to $\mathbb{Q}[\sqrt{a}]$ for some $a \in \mathbb{Z}$.