Class assignment: Noetherian rings

Rules: This is a test to be done in class during your monitor session. Please solve it in class and give to your monitor Renan Santos. Your solutions will have little influence on the final grade. If you use some result, please state it and (unless it's trivial) explain the proof.

Exercise 12.1. Let A be a Noetherian ring, and $B \subset A$ a subring. Prove that B is Noetherian, or find a counterexample.

Exercise 12.2. Let R be a Noetherian ring, and $S \subset R$ a multiplicatively closed subset. Prove that the localization $R[S^{-1}]$ is Noetherian.

Exercise 12.3 (2 points). Find a non-Noetherian ring where all prime ideals are maximal.

Exercise 12.4. Let A be the ring of holomorphic functions on $\mathbb{C}$. Prove that A is not Noetherian.

Exercise 12.5. Let $\mathfrak{a}_1, ..., \mathfrak{a}_n$ be ideals in a ring A such that $\bigcap_i \mathfrak{a}_i = 0$, and the quotients $A/\mathfrak{a}_i$ are Noetherian. Prove that A is Noetherian.

Exercise 12.6. Find a non-Noetherian ring R such that all maximal ideals in R are finitely generated.

Exercise 12.7. Let R be a non-Noetherian ring.

a. Prove that there exists an ideal $I \subset R$ which is not finitely-generated, but any ideal strictly containing I is finitely-generated.

b. Prove that such an ideal is always prime.