Geometria Algébrica I: final exam

Rules: Every student receives from me a list of 10 exercises (chosen randomly), and has to solve as many of them as you can by due date. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solutions, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes). Please contact me by email `verbit[]impa.br` when you are ready.

The final score N is obtained by summing up the points from the exam problems and the course assignments.[1] Marks: C when $40 \leqslant N < 60$, B when $60 \leqslant N < 80$, A when $80 \leqslant N < 110$, A+ when $N \geqslant 110$.

Remark 0.1. All ring are assumed to be Noetherian, all fields of characteristic 0, unless indicated otherwise.

1 Hilbert's Nullstellensatz and Zariski topology

Exercise 1.1 (10 points). Let $P(t_1, ..., t_n) \in \mathbb{R}[t_1, ..., t_n]$. Prove that the equation $P(t_1, ..., t_n) = 0$ has solutions in $\mathbb{R}^n$ if and only if the ring $A := \mathbb{R}[t_1, ..., t_n]/(P)$ contains a maximal ideal I such that $A/I \cong \mathbb{R}$.

Definition 1.1. **Boolean ring** is a ring such that all its elements are idempotents, that is, satisfy $a^2 = a$.

Exercise 1.2 (10 points). Prove that all prime ideals in a boolean ring are maximal, or find a counterexample.

Definition 1.2. **Spectrum** of a ring A is the set of all prime ideals of A. **Zariski topology** on the spectrum is defined by the base $\{U_f\}$ of open sets, enumerated by $f \in A$, defined as follows: a prime ideal $\mathfrak{p}$ belongs to U_f if $f \notin \mathfrak{p}$. **Maximal spectrum** is the set of maximal ideals with the same topology.

Exercise 1.3 (20 points). Prove that the spectrum of a boolean ring with Zariski topology is Hausdorff.

Exercise 1.4. Let M be a compact, connected manifold of positive dimension, $A = C(M)$ the ring of continuous functions on M, and $\mathfrak{p} \subset A$ an ideal.

a. (10 points) Prove that all $f \in \mathfrak{p}$ have a common zero in M.

b. (20 points) Prove that the spectrum of A has the same cardinality as continuum.

c. (20 points) Prove that any prime ideal of A is maximal, or find a counterxample.

Exercise 1.5 (20 points). Let $R := \frac{\mathbb{R}[x,y]}{(x^2+y^2-1)}$, be the ring of functions on the circle $S^1 \subset \mathbb{R}^2$ obtained by restrictions of polynomials. Prove that all ideals in R are principal, or find a counterexample.

Exercise 1.6 (20 points). Let $P(x, y)$ be an irreducible polynomial, and $R := \frac{\mathbb{C}[x,y]}{(P)}$. Prove that all ideals in R are principal, or find a counterexample.

Exercise 1.7 (10 points). Let $A = \mathbb{R}[t]$, and X its maximal spectrum. Prove that X is homeomorphic to the maximal spectrum of $\mathbb{C}[t]$.

[1] `http://www.verbit.ru/IMPA/AG-2018/res-scores.txt`

2 Categories, group representations, Noetherian rings

Remark 2.1. In this subsection, the rings are not finitely-generated or Noetherian unless noted otherwise.

Definition 2.1. A module is **finitely representable** if it is a quotient of a free module by a finitely generated module.

Exercise 2.1 (10 points). Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of A-modules, and M_1, M_3 are finitely representable. Prove that M_2 is also finitely representable.

Definition 2.2. A prime ideal is called **minimal** if it does not contain any smaller prime ideals.

Exercise 2.2 (10 points). Let A be a finitely generated ring over $\mathbb{C}$. Prove that the number of minimal prime ideals in A is finite.

Exercise 2.3 (20 points). Let C be a smooth 1-dimensional affine variety, equipped with an action of a finite group G. Prove that C/G is also smooth.

Exercise 2.4 (10 points). Let A be a ring without zero divisors. Suppose that any descending chain of ideals in A stabilizes. Prove that A is a field.

Exercise 2.5 (10 points). Prove that the category of finite groups is not equivalent to the category of finite abelian groups.

Exercise 2.6 (20 points). Let $\mathcal{C}_1$ be the category of finite-dimensional complex representations of the group $\mathbb{Z}/p\mathbb{Z}$, and $\mathcal{C}_2$ the category of all finite-dimensional complex representations of $\mathbb{Z}/q\mathbb{Z}$, where $p \neq q$ are prime numbers. Prove that the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are non-equivalent, or find a counterexample.

Exercise 2.7 (20 points). Let $\mathcal{C}_1$ be the category of finite-dimensional complex representations of the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{C}_2$ the category of all finite-dimensional complex representations of $\mathbb{Z}/4\mathbb{Z}$. Are the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ equivalent?

Exercise 2.8 (20 points). Let $\mathcal{C}$ be the category of vector spaces, and $F: \mathcal{C} \longrightarrow \mathcal{C}$ be a functor. Prove that $\dim F(V) \geqslant \dim V$ for any vector space V, unless $F(V) = 0$ for all V.

3 Tensor product

Definition 3.1. A module M over a ring R is called **flat** if the functor $X \longrightarrow X \otimes_R M$ is exact.

Definition 3.2. Let A be a ring without zero divisors. **Torsion** in an A-module M is the kernel of a natural map $M \longrightarrow M \otimes_A k(A)$.

Exercise 3.1 (10 points). Let M be a flat R-module, where R is a ring without zero divisors. Prove that M is torsion-free.

Exercise 3.2 (10 points). Let $M_1 \subset M_2 \subset ...$ be a sequence of embedded R-modules. Assume that all M_i are flat. Prove that $\bigcup M_i$ is also flat.

Exercise 3.3 (10 points). Prove that a finitely generated module over a local ring is flat if and only if it is free.

Exercise 3.4 (10 points). Let R_1, R_2 be rings over $\mathbb{C}$ such that $R_1 \otimes_{\mathbb{C}} R_2$ is finitely generated. Prove that R_1, R_2 are finitely generated, or find a counterexample.

Definition 3.3. An R-module M is called **invertible** if $M \otimes_R M^* \cong R$, where $M^* = \operatorname{Hom}_R(M, R)$.

Exercise 3.5 (10 points). Prove that each finitely generated, invertible R-module admits a monomorphism to a free R-module.

Exercise 3.6 (10 points). Let P be the set of isomorphism classes of invertible R-modules. Prove that tensor product defines a group structire on P (this group is called **Picard group** of P). Find P for a ring $\mathbb{C}[t]$.

Exercise 3.7 (20 points). Let X be a smooth affine variety of dimension 1, and $I \subset \mathcal{O}_X$ the maximal ideal of a point $x \in X$. Prove that I is invertible as an A-module.

Exercise 3.8 (10 points). Let M, N be finitely generated modules over the ring $R = \mathbb{C}[[t_1, ..., t_n]]$ of power series, and $M \otimes_R N = 0$. Prove that either $M = 0$ or $N = 0$.

Definition 3.4. Rank of a tensor $\psi \in V^{\otimes n}$ is a smallest number of tensor monomials $v_1 \otimes v_2 \otimes ... \otimes v_n$ such that ψ can be represented by a sum of these monomials.

Exercise 3.9 (10 points). Let $\dim V = d$. Prove that the maximal rank of a tensor $\psi \in V^{\otimes n}$ is d^{n-1}, or find a counterexample.

Exercise 3.10 (20 points). An R-module is called **projective** if it is a direct summand of a free R-module. Suppose that all finitely generated R-modules are projective. Prove that R is a direct sum of fields, or find a counterexample.

4 Normality, irreducibility, smoothness

Exercise 4.1 (10 points). Let A be the ring of complex-analytic functions on $\mathbb{C}$, and $k(A)$ its fraction field. Prove that the transcendence degree of A over $\mathbb{C}$ is infinite.

Exercise 4.2 (10 points). Let A be an integrally closed ring, G a finite group acting on A by automorphisms, and A^G the ring of invariants. Prove that A^G is integrally closed.

Exercise 4.3 (10 points). Prove that the ring $\mathbb{C}[x, y, z]/(y^2 - xz)$ is integrally closed.

Exercise 4.4 (20 points). Let $\zeta_n \in \mathbb{C}$ be a primitive root of unity of degree n. Define the action of the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ with coordinates x, y in such a way that the generator t maps x to $\zeta_n x$ and y to $\zeta_n^{-1} y$. Prove that the quotient $\mathbb{C}^2/G$ is singular.

Exercise 4.5 (20 points). Let G be the group of symmetries of a square, acting on $\mathbb{R}^2$. We induce the action of G on $\mathbb{C}^2 = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2$ in a natural way. Prove that $\mathbb{C}^2/G$ is smooth.

Exercise 4.6 (20 points). Let $A \subset \mathbb{C}^n$ be a subvariety given by equations $z_1^{p_1} = z_2^{p_2} = ... z_n^{p_n}$. Suppose that all p_i are different primes. Prove that A is irreducible.

Exercise 4.7 (10 points). Let $P(x, y, z)$ and $Q(x, y, z) \in \mathbb{C}[x, y, z]$ be irreducible, coprime polynomials, and X a variety defined by $P = Q = 0$. Prove that X is irreducible, or find a counterexample.

Exercise 4.8 (10 points). Let $S \subset \mathbb{C}^2$ be a smooth hypersurface defined by an irreducible quadratic equation. Prove that S is isomorphic to $\mathbb{C}$ or to $\mathbb{C}\backslash\{0\}$.

Exercise 4.9 (10 points). Let $S \subset \mathbb{C}^n$ be a smooth hypersurface given by a quadratic equation. Prove that S is normal.

5 Projective varieties and graded rings

Definition 5.1. Let $A_* = \bigoplus_{i \geqslant 0} A_i$ be a graded ring over $\mathbb{C}$. We say that A_* is of finite type if each graded component finite-dimensional, and $A_0 = \mathbb{C}$.

Exercise 5.1 (20 points). Let A be a finitely generated ring over $\mathbb{C}$ without zero divisors. Prove that A admits a finite type grading, or find a counterexample.

Exercise 5.2 (10 points). Let $\phi: \mathbb{C}P^2 \longrightarrow \mathbb{C}P^5$ be the Veronese embedding, Z its image, and $S \subset Z$ a 1-dimensional subvariety. Prove that there exists a hypersurface $V \subset \mathbb{C}P^5$ such that $V \cap Z = S$.

Exercise 5.3 (30 points). Let $\phi: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be an injective algebraic morphism. Prove that it is surjective, or find a counterexample.

Exercise 5.4 (10 points). Let $C \subset \mathbb{C}P^n$ be a projective variety birationally equivalent to $\mathbb{C}P^1$. Prove that C is isomorphic to $\mathbb{C}P^1$.

Exercise 5.5 (10 points). Let X, Y be affine varieties, $x \in X, y \in Y$ points. Consider the set Z obtained from X and Y by identifying x with y. Prove that there exists a structure of affine variety on Z such that the natural maps $X \longrightarrow Z$ and $Y \longrightarrow Z$ are affine morphisms.

Definition 5.2. Divisor in a variety X is a subvariety of codimension 1.

Exercise 5.6 (10 points). Let $D \subset \mathbb{C}P^n$ be a divisor. Prove that D can be defined by a single homogeneous equation.

Exercise 5.7 (10 points). Let $D_1, ..., D_m \subset \mathbb{C}P^n$ be a collection of divisors, with $m \leqslant n$. Prove that the intersection $D_1 \cap D_2 \cap ... \cap D_m$ is non-empty.

Exercise 5.8. Prove that the following 1-dimensional subvarieties ("plane curves") in $\mathbb{C}P^2$ are irreducible.

a. (10 points) A plane curve defined by $x^4y + y^4z + z^4x = 0$.

b. (10 points) $x^n + y^n + z^n = 0$.

c. (10 points) $x^2y^3 + y^2z^3 + z^2x^3 = 0$.