Geometria Algébrica I

lecture 4: Smooth manifolds

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Topological manifolds

REMARK: Manifolds can be smooth (of a given "differentiability class"), real analytic, or topological (continuous).

DEFINITION: **Topological manifold** is a topological space which is locally homeomorphic to an open ball in $\mathbb{R}^n$.

EXERCISE: **Show that a group of homeomorphisms acts on a connected manifold transitively.**

DEFINITION: Such a topological space is called **homogeneous**.

Open problem: (Busemann)
Characterize manifolds among other homogeneous topological spaces.

Now we whall proceed to the definition of smooth manifolds.

Banach fixed point theorem

LEMMA: (Banach fixed point theorem/"contraction principle")
Let $U \subset \mathbb{R}^n$ be a closed subset, and $f: U \longrightarrow U$ a map which satisfies $|f(x) - f(y)| < k|x - y|$, where $k < 1$ is a real number (such a map is called **"contraction"**). **Then f has a fixed point, which is unique.**

Proof. Step1: Uniqueness is clear because for two fixed points x_1 and x_2 $|f(x_1) - f(x_2)| = |x_1 - x_2| < k|x_1 - x_2|$.

Step 2: Existence follows because the sequence $x_0 = x, x_1 = f(x), x_2 = f(f(x)), ...$ satisfies $|x_i - x_{i+1}| \leqslant k|x_{i-1} - x_i|$ which gives $|x_n - x_{n+1}| < k^n a$, where $a = |x - f(x)|$. Then $|x_n - x_{n+m}| < \sum_{i=0}^m k^{n+i} a \leqslant k^n \frac{1}{1-k} a$, hence $\{x_i\}$ is a Cauchy sequence, and converges to a limit y, which is unique.

Step 3: $f(y)$ is a limit of a sequence $f(x_0), f(x_1), ... f(x_i), ...$ which gives $y = f(y)$. ■

EXERCISE: Find a counterexample to this statement when U is open and not closed.

Differentiable maps

DEFINITION: Let $U, V \subset \mathbb{R}^n$ be open subsets. **An affine map** is a sum of linear map α and a constant map. Its **linear part** is α.

DEFINITION: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets. A map $f: U \longrightarrow V$ is called **differentiable** if it can be approximated by an affine one at any point: that is, for any $x \in U$, there exists an affine map $\varphi_x: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that

$$\lim_{x_1 \to x} \frac{|f(x_1) - \varphi(x_1)|}{|x - x_1|} = 0$$

DEFINITION: **Differential**, or **derivative** of a differentiable map $f: U \longrightarrow V$ is the linear part of φ.

DEFINITION: **Diffeomorphism** is a differentiable map f which is invertible, and such that f^{-1} is also differentiable. A map $f: U \longrightarrow V$ is a **local diffeomorphism** if each point $x \in U$ has an open neighbourhood $U_1 \ni x$ such that $f: U_1 \longrightarrow f(U_1)$ is a diffeomorphism.

REMARK: **Chain rule** says that a composition of two differentiable functions is differentiable, and its differential is composition of their differentials.

REMARK: Chain rule implies that **differential of a diffeomorphism is invertible.** Converse is also true:

Inverse function theorem

THEOREM: Let $U, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a differentiable map. Suppose that the differential of f is everywhere invertible. **Then f is locally a diffeomorphism.**

Proof. Step1: Let $x \in U$. Without restricting generality, we may assume that $x = 0$, $U = B_r(0)$ is an open ball of radius r, and **in U one has $\frac{|f(x_1)-\varphi(x_1)|}{|x-x_1|} < 1/2$.** Replacing f with $-f \circ (D_0 f)^{-1}$, where $D_0 f$ is differential of f in 0, **we may assume also that $D_0 f = -\operatorname{Id}$.**

Step 2: In these assumptions, $|f(x)+x| < 1/2|x|$, hence $\psi_s(x) := f(x)+x-s$ is a contraction. This map maps $\overline{B}_{r/2}(0)$ to itself when $s < r/4$. By Banach fixed point theorem, **$\psi_s(x) = x$ has a unique fixed point x_s, which is obtained as a solution of the equation $f(x)+x-s = x$, or, equivalently, $f(x) = s$.** Denote the map $s \longrightarrow x_s$ by g.

Step 3: By construction, $fg = \operatorname{Id}$. Applying the chain rule again, we find that g is also differentiable. ■

REMARK: Usually in this course, diffeomorphisms would be assumed smooth (infinitely differentiable). **A smooth version of this result is left as an exercise.**

Critical points and critical values

DEFINITION: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a smooth function. A point $x \in U$ is a **critical point** of f if the differential $D_x f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is not surjective. **Critical value** is an image of a critical point. **Regular value** is a point of V which is not a critical value.

THEOREM: **(Sard's theorem) The set of critical values of f is of measure 0 in V.**

REMARK: We leave this theorem without a proof. We won't use it much.

DEFINITION: A subset $M \subset \mathbb{R}^n$ is **an m-dimensional smooth submanifold** if for each $x \in M$ there exists an open in $\mathbb{R}^n$ neighbourhood $U \ni x$ and a diffeomorphism from U to an open ball $B \subset \mathbb{R}^n$ which maps $U \cap M$ to an intersection $B \cap R^m$ of B and an m-dimensional linear subspace.

REMARK: Clearly, **a smooth submanifold is a (topological) manifold.**

THEOREM: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, $f: U \longrightarrow V$ a smooth function, and $y \in V$ a regular value of f. **Then $f^{-1}(y)$ is a smooth submanifold of U.**

Preimage of a regular value

THEOREM: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, $f: U \longrightarrow V$ a smooth function, and $y \in V$ a regular value of f. **Then $f^{-1}(y)$ is a smooth submanifold of U.**

Proof: Let $x \in U$ be a point in $f^{-1}(y)$. It suffices to prove that x has a neighbourhood diffeomorphic to an open ball B, such that $f^{-1}(y)$ corresponds to a linear subspace in B. Without restricting generality, we may assume that $y = 0$ and $x = 0$.

The differential $D_0 f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is surjective. Let $L := \ker D_0 f$, and let $A: \mathbb{R}^n \longrightarrow L$ be any map which acts on L as identity. Then $D_0 f \oplus A: \mathbb{R}^n \longrightarrow \mathbb{R}^m \oplus L$ is an isomorphism of vector spaces. **Therefore, $\Psi: f \oplus A$ mapping x_1 to $f(x_1) \oplus A(x_1)$ is a diffeomorphism in a neighbourhood of x.** However, $f^{-1}(0) = \Psi^{-1}(0 \oplus L)$. We have constructed a diffeomorphism of a neighbourhood of x with an open ball mapping $f^{-1}0)$ to $0 \oplus L$. ■

Preimage of a regular value: corollaries

COROLLARY: Let $f_1, ..., f_m$ be smooth functions on $U \subset \mathbb{R}^n$ such that the differentials df_i are linearly independent everywhere. **Then the set of solutions of equations $f_1(z) = f_2(z) = ... = f_m(z) = 0$ is a smooth $(n-m)$-dimensional submanifold in U.**

DEFINITION: **Smooth hypersurface** is a closed codimension 1 submanifold.

EXERCISE: Prove that **a smooth hypersurface in U is always obtained as a solution of an equation $f(z) = 0$,** where 0 is a regular value of a function $f: U \longrightarrow \mathbb{R}$.

Abstract manifolds: charts and atlases

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a **refinement** of a cover $\{U_i\}$ if every V_i is contained in some U_i.

REMARK: Any two covers $\{U_i\}$, $\{V_i\}$ of a topological space admit a common refinement $\{U_i \cap V_j\}$.

DEFINITION: Let M be a topological manifold. A cover $\{U_i\}$ of M is an **atlas** if for every U_i, we have a map $\varphi_i : U_i \to \mathbb{R}^n$ giving a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. In this case, one defines the **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

DEFINITION: A function $\mathbb{R} \longrightarrow \mathbb{R}$ is **of differentiability class C^i** if it is i times differentiable, and its i-th derivative is continuous. A map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **of differentiability class C^i** if all its coordinate components are. A **smooth function/map** is a function/map of class $C^\infty = \bigcap C^i$.

DEFINITION: An atlas is **smooth** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), **smooth of class C^i** if all transition functions are of differentiability class C^i, and **real analytic** if all transition maps admit a Taylor expansion at each point.

Smooth structures

DEFINITION: A **refinement** of an **atlas** is a refinement of the corresponding cover $V_i \subset U_i$ equipped with the maps $\varphi_i : V_i \to \mathbb{R}^n$ that are the restrictions of $\varphi_i : U_i \to \mathbb{R}^n$. Two atlases (U_i, φ_i) and (U_i, ψ_i) of class C^∞ or C^i (with the same cover) are **equivalent** in this class if, for all i, the map $\psi_i \circ \varphi_i^{-1}$ defined on the corresponding open subset in $\mathbb{R}^n$ belongs to the mentioned class. Two arbitrary atlases are **equivalent** if the corresponding covers possess a common refinement.

DEFINITION: A **smooth structure** on a manifold (of class C^∞ or C^i) is an atlas of class C^∞ or C^i considered up to the above equivalence. A **smooth manifold** is a topological manifold equipped with a smooth structure.

DEFINITION: A **smooth function** on a manifold M is a function f whose restriction to the chart (U_i, φ_i) gives a smooth function $f \circ \varphi_i^{-1} : \varphi_i(U_i) \longrightarrow \mathbb{R}$ for each open subset $\varphi_i(U_i) \subset \mathbb{R}^n$.

Smooth maps and isomorphisms

From now on, **I shall identify the charts U_i with the corresponding subsets of $\mathbb{R}^n$,** and forget the differentiability class.

DEFINITION: **A smooth map** of $U \subset \mathbb{R}^n$ to a manifold N is a map $f: U \longrightarrow N$ such that for each chart $U_i \subset N$, the restriction $f\big|_{f^{-1}(U_i)}: f^{-1}(U_i) \longrightarrow U_i$ is smooth with respect to coordinates on U_i. **A map of manifolds** $f: M \longrightarrow N$ is **smooth** if for any chart V_i on M, the restriction $f\big|_{V_i}: V_i \longrightarrow N$ is smooth as a map of $V_i \subset \mathbb{R}^n$ to N.

DEFINITION: **An isomorphism** of smooth manifolds is a bijective smooth map $f: M \longrightarrow N$ such that f^{-1} is also smooth.

Sheaves

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and exact sequences

REMARK: A presheaf of functions is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions is a presheaf allowing "gluing"** a function on a bigger open set if its restrictions to smaller open sets are compatible.

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\Big|_{U_i \cap U_j}$ and $-\eta\Big|_{U_j \cap U_i}$.

Sheaves and presheaves: examples

Examples of sheaves:

* Space of continuous functions

* Space of smooth functions, any differentiability class

* Space of real analytic functions

Examples of presheaves which are not sheaves:

* Space of constant functions **(why?)**

* Space of bounded functions **(why?)**

Ringed spaces

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

REMARK: Let $f: X \longrightarrow Y$ be a smooth map of smooth manifolds. Since a pullback $f^*\mu$ of a smooth function $\mu \in C^\infty(M)$ is smooth, **a smooth map of smooth manifolds defines a morphism of ringed spaces.**

Converse is also true:

Ringed spaces and smooth maps

CLAIM: Let (M, C^i) and (N, C^i) be manifolds of class C^i. Then **there is a bijection between smooth maps $f: M \longrightarrow N$ and the morphisms of corresponding ringed spaces.**

Proof: Any smooth map induces a morphism of ringed spaces. Indeed, **a composition of smooth functions is smooth, hence a pullback is also smooth.**

Conversely, let $U_i \longrightarrow V_i$ be a restriction of f to some charts; to show that f is smooth, it would suffice to show that $U_i \longrightarrow V_i$ is smooth. However, we know that a pullback of any smooth function is smooth. **Therefore, Claim is implied by the following lemma.**

LEMMA: Let M, N be open subsets in $\mathbb{R}^n$ and let $f: M \to N$ map such that a pullback of any function of class C^i belongs to C^i. **Then f is of class C^i.**

Proof: Apply f to coordinate functions. ■

A new definition of a manifold

As we have just shown, this definition is equivalent to the previous one.

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{B}^n$ of this class, where $\mathbb{B}^n$ denotes an open ball in $\mathbb{R}^n$.

DEFINITION: **A chart**, or **a coordinate system** on an open subset U of a manifold $(M, \mathcal{F})$ is an isomorphism between $(U, \mathcal{F})$ and an open subset in $(\mathbb{B}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions of the same class on $\mathbb{B}^n$.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphim of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.