Geometria Algébrica I

lecture 6: Representations of finite groups

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Group representations

DEFINITION: Representation of a group G is a homomorphism $G \longrightarrow GL(V)$. In this case, V is called **representation space**, and **a representation**.

DEFINITION: Irreducible representation is a representation having no G-invariant subspaces. **Semisimple representation** is a direct sum of irreducible ones.

REMARK: If the group G acts on a vector space V, it also acts on all tensor powers of V (action os extended by multiplicativity). In particular, **G acts on $V^* \otimes V^*$ as $g(h)(x,y) = h(g(x), g(y))$**, for any $g \in G$, $h \in V^* \otimes V^*$ and $x, y \in V$.

DEFINITION: A metric h (Euclidean or Hermitian) on a vector space V is called **G-invariant** if the corresponding tensor $h \in V^* \otimes_{\mathbb{R}} V^*$ is G-invariant.

G-invariant metrics

CLAIM:
A sum of two Hermitian (Euclidean) metrics is Hermitian (Euclidean).
■

COROLLARY: Let V be a representation of a finite group (over $\mathbb{R}$ or $\mathbb{C}$). **Then V admits a G-invariant metric** (Hermitian or Euclidean).

Proof: Let h be an arbitrary metric, and $\frac{1}{|G|}\sum_{g\in G} g(h)$ its average over the G action. The previous claim implies that it is a metric. Since G acts on itself bijectively, interchanging all terms in the sum, **it is G-invariant.** ■

COROLLARY: Let $E \subset V$ be a subrepresentation in a finite group representation over $\mathbb{R}$ or $\mathbb{C}$. Then **V can be decomposed onto a direct sum of two G-representations $V = W \oplus W'$.**

Proof: Choose a G-invariant metric on V, and let $W^\perp$ be the orthogonal complement to W. Then $W^\perp$ is also G-invariant **(check this)**. This gives a decomposition $V = W \oplus W^\perp$. ■

COROLLARY: Any finite-dimensional representation of a finite group is semisimple. ■

Exact functors

DEFINITION: An **exact sequence** is a sequence of vector spaces and maps $... \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ such the kernel of each map is the image of the previous one. A **short exact sequence** is exact sequence of form $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$. **Here "exact" means that i is injective, j surjective, and image of i is kernel of j.**

DEFINITION: A functor $A \longrightarrow FA$ on the category of R-modules or vector spaces is called **left exact** if any exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is mapped to an exact sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC,$$

right exact if it is mapped to an exact sequence

$$FA \longrightarrow FB \longrightarrow FC \longrightarrow 0,$$

and **exact** if the sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

is exact.

Invariants and coinvariants

DEFINITION: Let G be a finite group, and V its representation. Define **the space of G-invariants** V^G as the space of all G-invariant vectors, and **the space of coinvariants** as the quotient of V by its subspace generated by vectors $v - g(v)$, where $g \in G, v \in V$.

CLAIM: Let V be an irreducible representation of G. **Then its invariants and co-ivariants are equal 0 if it is non-trivial, and equal V if it is trivial.**

COROLLARY: Let V be a semisimple representation of G. **Then $V_G = V^G$.**

EXERCISE: Prove that **the functor $V \longrightarrow V^G$ is left exact, and $V \longrightarrow V_G$ is right exact**.

COROLLARY: For any finite group G, the functor of G-invariants is exact.

REMARK: The averaging map

$$m \longrightarrow \frac{1}{|G|} \sum_{g \in G} g(m)$$

gives a projection of V to V^G, and the kernel of this map is the kernel of the natural projection $V \longrightarrow V_G$

Ring of invariants and quotient space

DEFINITION: Action of a group G on an affine manifold A is the action of G on the ring $\mathcal{O}_A$ of polynomial functions on A.

REMARK: By Strong Nullstellensatz, **this is the same as action of G on A by automorphisms.**

REMARK: We want to define the quotient space A/G as the algebraic variety associated with the invariant ring $\mathcal{O}_A^G$.

Problem # 1: We need to show that the ring $\mathcal{O}_A^G$ is finitely generated (Noether theorem).

Problem # 2: We need to identify the maximal ideals in $\mathcal{O}_A^G$ with the elements of the quotient set A/G.

Noether theorem (scheme of the proof)

THEOREM: Let R be a finitely generated ring over $\mathbb{C}$, and G a finite group acting on R by automorphisms. Then **the ring R^G of G-invariants is finitely generated**.

Scheme of the proof:

1. Noetheriannes of R is used to prove that R^G is Noetherian.

2. Prove that R^G is finite generated for $R = \mathbb{C}[z_1, ..., z_n]$, where R acts on polynomials of degree 1 by linear automorphisms.

3. Deduce the general case from (2) and exactness of $V \longrightarrow V^G$

Ideals in R and R^G

LEMMA: Let R be a ring, G a finite group acting on R, R^G the ring of G-invariants, and $I \subset R^G$ an ideal. Then **ideal RI satisfies $\operatorname{Av}_G(RI) = \operatorname{Av}_G(R)I = R^G I = I$,** where $\operatorname{Av}_G : R \longrightarrow R^G$ denotes the averaging map. ■

COROLLARY: Let $I_1 \subsetneq I$ be ideals in R^G. **Then $RI_1 \subsetneq RI$.** ■

COROLLARY 1: In these assumptions, **if R is Noetherian, then R^G is also Noetherian.**

Proof: Any infinite, strictly monotonous sequence $I_0 \subsetneq I_1 \subsetneq ...$ of ideals in R^G gives a strictly monotonous sequence $RI_0 \subsetneq RI_1 \subsetneq ...$ in R. ■

Graded rings

DEFINITION: A graded ring is a ring A^*, $A^* = \bigoplus_{i=0}^{\infty} A^i$, with multiplication which satisfies $A^i \cdot A^j \subset A^{i+j}$ ("grading is multiplicative"). A graded ring is called **of finite type** if all A^i are finitely dimensional.

EXAMPLE: Polynomial ring $\mathbb{C}[V] = \bigoplus_i \operatorname{Sym}^i V$ is clearly graded.

Graded rings (2)

Claim 1: Let A^* be a graded ring of finite type. Then **A^* is Noetherian $\Leftrightarrow$ it is finitely generated**.

Proof. Step1: If A^* is finitely generated, it is Noetherian by Hilbert's basis theorem.

Step 2: Conversely, suppose that A^* is Noetherian. Then the ideal $\bigoplus_{i>0} A^i \subset A^*$ is finitely generated. Let $a_i \in A^{n_i}$ be generators of this ideal over A^*. **We are going to show that products of a_i generate A^*.**

Step 3: Let $z \in A^*$ be a graded element of smallest degree which is not generated by products of a_i. Since a_i generate the ideal $\bigoplus_{i>0} A^i \subset A^*$, we can express z as $z = \sum_i f_i a_i$, where $f_i \in A^*$. However, $\deg f_i < \deg z$, hence all f_i are generated by products of a_i. Then all f_i are generated by products of a_i.
■

A caution: In this argument, two notions of "finitely generated" are present: finitely generated ideals (an additive notion) and finitely generated rings over $\mathbb{C}$ (multiplicative). **These two notions are completely different!** One is defined for ideals (or R-modules), another for a ring over a field. Only the name is the same (bad terminology).

Proof of Noether theorem for polynomial invariants

DEFINITION: Let V be a vector space with basis $z_1, ..., z_n$, and $\mathbb{C}[V] = \bigoplus_i \operatorname{Sym}^i V = \mathbb{C}[z_1, ..., z_n]$ the corresponding polynomial ring. Suppose that G acts on V by linear automorphisms. We extend this action to the symmetric tensors $\bigoplus_i \operatorname{Sym}^i V$ multiplicatively. This implies that G acts on $\mathbb{C}[V]$ by automorphisms. Such action is called **linear**.

CLAIM: (Noether theorem for polynomial invariants)
Let G act linearly on the polynomial ring $\mathbb{C}[V]$. **Then the invariant ring $\mathbb{C}[V]^G$ is finitely generated.**

Proof. Step1: Since the action of G preserves the grading on $\mathbb{C}[V]$, **the ring $\mathbb{C}[V]^G$ is graded and of finite type.**

Step 2: $\mathbb{C}[V]^G$ **is Noetherian,** because $\mathbb{C}[V]$ is Noetherian, and invariant rings in Noetherian rings are Noetherian (Corollary 1).

Step 3: A finite type Noetherian graded ring is finitely generated by Claim 1. ■

Noether theorem

THEOREM: (Noether theorem)
Let R be a finitely generated ring over $\mathbb{C}$, and G a finite group acting on R by automorphisms. Then **the ring R^G of G-invariants is finitely generated**.

Proof. Step1: Let $f_1, ..., f_m$ be generators of R, and $\{g_1, ..., g_k\} = G$. Consider the space $V \subset R$ generated by all vectors $g_i f_j$. Clearly, $V \subset R$ is V-invariant, and **the natural homomorphism $\mathbb{C}[V] \longrightarrow R = \mathbb{C}[V]/I$ is surjective and G-invariant.**

Step 2: The natural map $\mathbb{C}[V]^G \longrightarrow R^G$ is surjective, because the functor $W \longrightarrow W^G$ is exact.

Step 3: The ring $\mathbb{C}[V]^G$ is finitely generated by Noether theorem for polynomial invariants, hence its quotient R^G is also finitely generated. ■