Geometria Algébrica I

lecture 7: Tensor product

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Tensor product

DEFINITION: Let R be a ring, and M, M' modules over R. We denote by $M \otimes_R M'$ an R-module generated by symbols $m \otimes m'$, $m \in M, m' \in M'$, modulo relations
$r(m \otimes m') = (rm) \otimes m' = m \otimes (rm')$,
$(m + m_1) \otimes m' = m \otimes m' + m_1 \otimes m'$,
$m \otimes (m' + m'_1) = m \otimes m' + m \otimes m'_1$ for all $r \in R, m, m_1 \in M, m', m'_1 \in M'$. Such an R-module is called **the tensor product of M and M' over R.**

REMARK: Suppose that M is generated over R by a set $\{m_i \in M\}$, and M' generated by $\{m'_j \in M'\}$. **Then $M \otimes_R M'$ is generated by $\{m_i \otimes m'_j\}$.**

EXERCISE: Find two non-zero R-modules A, B such that $A \otimes_R B = 0$ when

a. $R = \mathbb{Z}$.

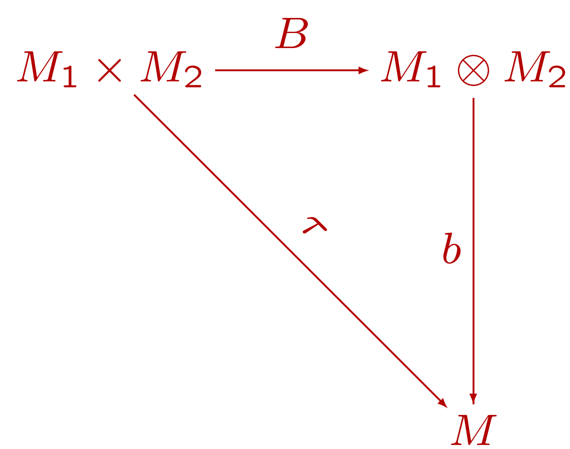
b. $R = C^\infty M$ the ring of smooth functions on a manifold.

c. $R = \mathbb{C}[t]$ (polynomial ring).

Bilinear maps

DEFINITION: Let M_1, M_2, M be modules over a ring R. **Bilinear map** $\mu(M_1, M_2) \stackrel{\varphi}{\longrightarrow} M$ is a map satisfying $\varphi(rm, m') = \varphi(m, rm') = r\varphi(m, m')$, $\varphi(m + m_1, m') = \varphi(m, m') + \varphi(m_1, m')$, $\varphi(m, m' + m'_1) = \varphi(m, m') + \varphi(m, m'_1)$.

THEOREM: (Universal property of the tensor product)
For any bilinear map $B: M_1 \times M_2 \longrightarrow M$ **there exists a unique homomorphism $b: M_1 \otimes M_2 \longrightarrow M$, making the following diagram commutative:**



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REMARK: If R is the field k, R-modules are vector spaces, and the previous theorem proves that $\mathsf{Bil}(M_1 \times M_2, k) = (M_1 \otimes M_2)^*$. For finite-dimensional M_i, it gives $M_1 \otimes M_2 = (M_1 \otimes M_2)^{**} = \mathsf{Bil}(M_1 \times M_2, k)^*$.

Universal property of the tensor product and categories

DEFINITION: Initial object of a category $\mathcal{C}$ is an object $X \in \mathcal{O}b(\mathcal{C})$ such that for any $Y \in \mathcal{O}b(\mathcal{C})$ there exists a unique morphism $X \longrightarrow Y$.

EXAMPLE: Zero space is an initial object in the category of vector spaces. The ring $\mathbb{Z}$ is an initial object in the category of rings with unit.

EXERCISE: Prove that **initial object is unique.**

DEFINITION: Let M_1, M_2 are R-modules, and $\mathcal{C}$ the following category. Objects of $\mathcal{C}$ are pairs (R-module M, bilinear map $M_1 \times M_2 \longrightarrow M$). Morphisms of $\mathcal{C}$ are homomorphisms $M \xrightarrow{\varphi} M'$ making the following diagram commutative:

$$\begin{array}{ccc} M_1 \times M_2 & \longrightarrow & M \\ {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ M_1 \times M_2 & \longrightarrow & M' \end{array}$$

CLAIM: (Universal property of the tensor product)
Tensor product $M_1 \times M_2$ is the initial object in $\mathcal{C}$.

COROLLARY: Tensor product is uniquely determined by the universal property.

Indeed, the initial object is unique.

The internal $\mathcal{H}om$ and exact functors

DEFINITION: Let M, M' be R-modules. Consider the group $\operatorname{Hom}_R(M, M')$ of R-module homomorphisms. We consider $\operatorname{Hom}_R(M, M')$ as an R-module, using $r\varphi(m) := \varphi(rm)$. This R-module is called **internal** Hom **functor**, denoted $\mathcal{H}om$.

Claim 1: Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. **Then the natural sequences**

$$0 \longrightarrow \mathcal{H}om_R(M_3, N) \longrightarrow \mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N)$$

and

$$0 \longrightarrow \mathcal{H}om_R(N, M_1) \longrightarrow \mathcal{H}om_R(N, M_2) \longrightarrow \mathcal{H}om_R(N, M_3)$$

are exact, for any R-module M.

Proof: Let's prove exactness of the first sequence. Exactness in the term $\mathcal{H}om_R(M_3, N)$ is clear. If $\nu \in \mathcal{H}om_R(M_2, N)$ is mapped to 0 in projection to $\mathcal{H}om_R(M_1, N)$, this means that $\nu|_{M_1} = 0$, giving a morphism $\tilde{\nu} \in \mathcal{H}om_R(M_3, N)$, which is mapped to ν. **Exactness of the second sequence is left as an exercise.** ■

The internal $\mathcal{H}om$ and tensor product

REMARK: Universal property of $\otimes_R$ implies

$$\mathcal{H}om_R(M_1 \otimes_R M_2, M) = \mathcal{H}om_R(M_1, \mathcal{H}om_R(M_2, M)).$$

Indeed, **the group $\mathcal{H}om_R(M_1, \mathcal{H}om_R(M_2, M))$ is identified with the group of bilinear maps $M_1 \times M_2 \longrightarrow M$.**

COROLLARY: Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. **Then for any R-modules N, N', the sequence**

$$0 \longrightarrow \mathcal{H}om_R(M_3 \otimes N', N) \longrightarrow \mathcal{H}om_R(M_2 \otimes N', N) \longrightarrow \mathcal{H}om_R(M_1 \otimes N', N)$$

is exact.

Proof: Using Claim 1 twice, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow & \mathcal{H}om_R(N', \mathcal{H}om_R(M_3, N)) \\ & \longrightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_2, N)) \longrightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_3, N)). \end{aligned}$$

Then we use an isomorphism $\mathcal{H}om_R(A \otimes_R B, M) = \mathcal{H}om_R(A, \mathcal{H}om_R(B, M))$ proven above. ■

Functor $\mathcal{H}om$, part 2

REMARK: Exactness of the sequence $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ implies exactness of $0 \longrightarrow \mathcal{H}om_R(M_3, N) \longrightarrow \mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N)$. We are going to prove the converse: **exactness of the second sequence (for all N) implies exactness of the first one.**

DEFINITION: A complex of R-modules is a sequence $M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \xrightarrow{d_3} ...$ such that $d_i \circ d_{i+1} = 0$.

LEMMA: Consider a complex E^* of R-modules $M_1 \xrightarrow{\mu} M_2 \xrightarrow{\rho} M_3 \longrightarrow 0$ such that $0 \longrightarrow \mathcal{H}om_R(M_3, N) \xrightarrow{\rho_N} \mathcal{H}om_R(M_2, N) \xrightarrow{\mu_N} \mathcal{H}om_R(M_1, N)$ is exact for all N. **Then E^* is also exact.**

Proof: Injectivity of ρ_N implies surjectivity of ρ, if we put $N = M_3/\operatorname{im}\rho$. Exactness of the second sequence in term $\mathcal{H}om_R(M_2, N)$ implies exactness of E in term M_2 when $N = M_2/\operatorname{im}\mu$. ■

Exactness of the tensor product

THEOREM: Let $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. **Then the sequence**

$$M_1 \otimes_R M \longrightarrow M_2 \otimes_R M \longrightarrow M_3 \otimes_R M \longrightarrow 0 \qquad (*)$$

is exact.

Proof: Using the universal property of tensor product, we have shown that

$$0 \longrightarrow \mathcal{H}om_R(M_3 \otimes M, N) \longrightarrow \mathcal{H}om_R(M_2 \otimes M, N) \longrightarrow \mathcal{H}om_R(M_1 \otimes M, N)$$

is exact for any N. Aplying the previous lemma, we obtain that (*) is also exact. ■

COROLLARY: Let $I \subset R$ be an ideal in a ring. **Then $M \otimes_R (R/I) = M/IM$.**

Proof: Apply the functor $\otimes_R M$ to the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. We obtain $IM \longrightarrow M \longrightarrow (R/I) \otimes_R M \longrightarrow 0$. ■

Tensor product: examples

EXERCISE: Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$.

REMARK: Let $\mathbb{Z} \xrightarrow{\varphi} Z$ be a multiplication by 2. Then the sequence

$$\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

obtained from $0 \longrightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ by $\otimes_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ **is not left exact**.

Tensor product of rings: geometric meaning

EXERCISE: Let $f: X \longrightarrow Y$ be a morphism of algebraic varieties, $y \in Y$ a point. **Prove that $f^{-1}(y)$ is affine.**

QUESTION: How one describes the ring of regular functions on $f^{-1}(y)$?

HINT: Use the tensor product of rings!

EXERCISE: Let $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^k$ be algebraic subvarieties. **Prove that $X \times Y$ is also algebraic.**

HINT: Use the tensor product of rings!

Tensor product of rings

DEFINITION: Let A, B be rings, $C \longrightarrow A$, $C \longrightarrow B$ homomorphisms. Consider A and B as C-modules, and let $A \otimes_{\mathbb{C}} B$ be their tensor product. Define the ring multiplication on $A \otimes_C B$ as $a \otimes b \cdot a' \otimes b' = aa' \otimes bb'$. This defines **tensor product of rings**.

EXAMPLE: $\mathbb{C}[t_1, ..., t_k] \otimes_{\mathbb{C}} \mathbb{C}[z_1, ..., z_n] = \mathbb{C}[t_1, ..., t_k, z_1, ..., z_n]$. Indeed, if we denote by $\mathbb{C}_d[t_1, ..., t_k]$ the space of polynomials of degree d, then $\mathbb{C}_d[t_1, ..., t_k] \otimes_{\mathbb{C}} \mathbb{C}_{d'}[z_1, ..., z_n]$ is polynomials of degree d in $\{t_i\}$ and d' in $\{z_i\}$.

EXAMPLE: For any homomorphism $\varphi : \mathbb{C} \longrightarrow A$, **the ring $A \otimes_C (C/I)$ is a quotient of A by the ideal $A \cdot \varphi(I)$**. This follows from $M \otimes_R (R/I) = M/IM$.

PROPOSITION: (associativity of $\otimes$)
Let $C \longrightarrow A, C \longrightarrow B, C' \longrightarrow B, C' \longrightarrow D$ be ring homomorphisms. Then $(A \otimes_C B) \otimes_{C'} D = A \otimes_C (B \otimes_{C'} D)$.

Proof: Universal property of $\otimes$ implies that $\operatorname{Hom}((A \otimes_C B) \otimes_{C'} D, M) = \operatorname{Hom}(A \otimes_C (B \otimes_{C'} D), M)$ is the space of polylinear maps $A \otimes B \otimes D \longrightarrow M$ satisfying $\varphi(ca, b, d) = \varphi(a, cb, d)$ and $\varphi(a, c'b, d) = \varphi(a, b, c'd)$. However, an object X of category is defined by the functor $\operatorname{Hom}(X, \cdot)$ uniquely **(prove it)**.
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