Geometria Algébrica I

lecture 12: normalization

Misha Verbitsky

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Dominant morphisms (reminder)

DEFINITION: Zariski topology on an algebraic variety is a topology where the closed subsets are algebraic subsets. **Zariski closure** of $Z \subset M$ is an intersection of all Zariski closed subsets containing Z.

EXERCISE: Prove that **Zariski topology on $\mathbb{C}$ coincides with the cofinite topology.**

CAUTION: Zariski topology is non-Hausdorff.

DEFINITION: Dominant morphism is a morphism $f: X \longrightarrow Y$, such that Y is a Zariski closure of $f(X)$.

PROPOSITION: Let $f: X \longrightarrow Y$ be a morphism of affine varieties. **The morphism f is dominant if and only if the homomorphism $\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$ is injective**.

Field of fractions (reminder)

DEFINITION: Let $S \subset R$ be a subset of R, closed under multiplication and not containing 0. **Localization** of R in S is a ring, formally generated by symbols a/F, where $a \in R$, $F \in S$, and relations $a/F \cdot b/G = ab/FG$, $a/F + b/G = \frac{aG+bF}{FG}$ and $aF^k/F^{k+n} = a/F^n$.

DEFINITION: Let R be a ring without zero divisors, and S the set of all non-zero elements in R. **Field of fractions** of R is a localization of R in S.

CLAIM: Let $f: X \longrightarrow Y$ be a dominant morphism, where X is irreducible. **Then Y is also irreducible**. Moreover, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ **can be extended to a homomorphism of the fields of fractions. $k(Y) \longrightarrow k(X)$.**

DEFINITION: A dominant morphism of irreducible varieties is called **birational** if the corresponding homomorphism of the fields of fractions is an isomorphism.

Integral dependence (reminder)

DEFINITION: Let $A \subset B$ be rings. An element $b \in B$ is called **integral over A** if the subring $A[b] = A \cdot \langle 1, b, b^2, b^3, ...\rangle$, generated by b and A, is finitely generated as A-module.

DEFINITION: Monic polynomial is a polynomial with leading coefficient 1.

CLAIM: An element $x \in B$ **is integral over $A \subset B$ if and only if the chain of submodules**

$$A \subset A \cdot \langle 1, x\rangle \subset A \cdot \langle 1, x, x^2\rangle \subset A \cdot \langle 1, x, x^2, x^3\rangle \subset ...$$

terminates.

COROLLARY: An element $x \in B$ is integral over $A \subset B \Leftrightarrow x$ is a root of a monic polynomial with coefficients in A. ■

CLAIM: Let $A \subset B$ be Noetherian rings. Then **sum and product of elements which are integral over A is also integral.**

Integral closure (reminder)

DEFINITION: Let $A \subset B$ be rings. The set of all elements in B which are integral over A is called **the integral closure of A in B**.

DEFINITION: Let A be the ring without zero divisors, and $k(A)$ its field of fractions. The set of all elements $a \in k(A)$ which are integral over A is called **the integral closure of A**. A ring A is called **integrally closed** if A coincides with its interal closure in $k(A)$.

REMARK: As shown above, **the integral closure is a ring.**

DEFINITION: An affine variety X is called **normal** if all its irreducible components X_i are disconnected, and the ring of functions $\mathcal{O}_{X_i}$ for each of these irreducible components is integrally closed.

REMARK: Equivalently, **X is normal if any finite, birational morphism $Y \longrightarrow X$ is an isomorphism.**

Factorial rings

DEFINITION: An element p of a ring R is called **prime** if the corresponding principal ideal (p) is prime.

DEFINITION: A ring R without zero divisors is called **factorial** if any element $r \in R$ can be represented as a product of prime elements, $r = \prod_i p_i^{\alpha_i}$, and this decomposition is unique up to invertible factors and permutation of p_i.

PROPOSITION: Let A be a factorial ring. Then it is integrally closed.

Proof. Step1: Let $u, v \in A$, and $u/v \in k(A)$ a root of a monic polynomial $P(t) \in A[t]$ of degree n. **Then u^n is divisible by v in A.**

Step 2: Let $u/v \in k(A)$ be a root of a monic polynomial $P(t) \in A[t]$. Assume that u, v are comprime. **Since u^n is divisible by v, and they are coprime, v is invertible by factoriality of A.** Then $u/v \in A$. ■

Gauss lemma

EXERCISE: Let R be a ring without zero divisors. **Prove that the polynomial ring $R[t]$ has no zero divisors.**

THEOREM: ("Gauss lemma")
Let R be a factorial ring. **Then the ring of polynomials $R[t]$ is also factorial.**

Proof: See the next slide.

DEFINITION: Let R be a factorial ring. A polynomial $P(t) \in R[t]$ is called **primitive** if the greatest common divisor of its coefficients is 1.

Lemma 1: Let $P_1, P_2 \in R[t]$ be primitive polynomials. **Then their product is also primitive**.

Proof: Let $p \in R$ be a prime. Since the polynomials P_1, P_2 are primitive, they are non-zero modulo p. Since the ring $R/(p)$ has no zero divisors, **the product P_1P_2 is non-zero in $R/(p)[t]$,** hence the greatest common divisor of the coefficients of P_1P_2 is not divisible by p. ■

Irreducibility of polynomials in $R[t]$ and $K[t]$

Lemma 1: Let $P_1, P_2 \in R[t]$ be primitive polynomials. **Then their product is also primitive**.

Lemma 2: Let R be a factorial ring, and K its fraction field. **Then any primitive polynomial $P \in R[t]$, which is irreducible in $R[t]$, is also irreducible in $K[t]$.**

Proof: Assume that P is decomposable in $K[t]$. Then $rP = P_1P_2$, where $P_1, P_2 \in R[t]$ and $r \in R$. Let s_1, s_2 be the greatest common divisors of the coefficients of P_1, P_2. Then $rP = s_1s_2P_1'P_2'$, and P_1, P_2 are primitive. In this case P_1P_2 is primitive (Lemma 1), hence the greatest common divisor of the coefficients of $s_1s_2P_1'P_2'$ is s_1s_2. Since P is also primitive, the greatest common divisor of the coefficients of $rP = s_1s_2P_1'P_2'$ is r. **Then $\frac{r}{s_1s_2}$ is invertible, and P is decomposable in $R[t]$.** ■

Gauss lemma (proof)

THEOREM: ("Gauss lemma")
Let R be a factorial ring. **Then the ring of polynomials $R[t]$ is also factorial.**

Proof: Let K be the fraction field of R. The ring $K[t]$ is factorial, because it is Euclidean (handout 3). Lemma 2 implies that a prime decomposition of a primitive polynomial $P(t) \in R[t]$ is uniquely determined by its prime decomposition in $K[t]$, hence it is unique. A non-primitive polynomial is decomposed as a product of the greatest common divisor of its coefficients and a primitive polynomial, hence its prime decomposition is also unique. ■

COROLLARY: The affine space $\mathbb{C}^n$ is a normal variety. Moreover, **for any variety X with factorial ring $\mathcal{O}_X$ of regular functions, the product $X \times \mathbb{C}^n$ is also normal.**

Proof: As we have shown previously, $\mathcal{O}_{X\times\mathbb{C}^n} = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[t_1, ..., t_n] = \mathcal{O}_X[t_1, ..., t_n]$. This ring is factorial by Gauss lemma. ■

Finiteness of integral closure

THEOREM: Let A be an integrally closed Noetherian ring, $[K : k(A)]$ a finite extension of its field of fractions, and B the integral closure of A in K. **Then B is finitely generated as an A-module.**

Proof. Step1: For any $a \in B$, denote by $L_b : \ K \longrightarrow K$ the map of multiplication by b. Consider L_b as a $k(A)$-linear endomorphism of thefinite-dimensional space K over $k(A)$, and define **the trace** $\operatorname{Tr}(b) := \operatorname{Tr}(L_b)$. Since $\operatorname{Tr}(b) = \frac{d}{dt}\det(tId_K - tL_b)(0)$, for any $b \in B$ integral over A, **the trace of b is integral over A.**

Step 2: The bilinear symmetric form $x, y \longrightarrow \operatorname{Tr}(xy)$ is non-degenerate. Indeed, $\operatorname{Tr}(xx^{-1}) = \dim_{k(A)} K$, and $\operatorname{char} k(A) = 0$.

Step 3: Choose a basis $e_1, ..., e_n$ in the $k(A)$-vector space K. Let $P_i(t) \in k(A)[t]$ be the minimal polynomials of e_i. Write $P_i(t) = A_i t^{n_i} + \sum_{j<n_i} a_{ij} t^j$, where $A_i, a_{ij} \in A$. Then $A_i e_i$ is a root of a monic polynomial $\tilde{P}_i(t) = t^{n_i} + \sum_{j<n_i} A^{n_i - j} a_{ij} t^j$. **This proves that the basis $e_1, ..., e_n$ in $K : k(A)$ can be chosen such that all e_i are integral over A.**

Finiteness of integral closure (2)

Step 4: Let $e_i^* \in K$ be the dual basis with respect to the form Tr, with $\operatorname{Tr}(e_i^* e_j) = \delta_{ij}$. Consider the A-module $M \subset K$ generated by e_i^*. Clearly, $M := \{b \in K \mid \operatorname{Tr}(be_i) \in A\}$.

Step 5: For any $b \in B$, the trace $\operatorname{Tr}(b)$ belongs to A, because b is integral over A (Step 1). Then $B \subset M$, and **B is a submodule of a finitely generated A-module M.** Since A is Noetherian, B is finitely generated as A-module. ■

COROLLARY: Let B be a ring over $\mathbb{C}$. Assume that there exists an injective ring morphism from $A = \mathbb{C}[x_1, ..., x_k]$ to B such that B is finitely generated as an A-module. **Then its integral closure $\hat{B}$ is a finitely generated A-module**. In particlular, **$\hat{B}$ is a finitely generated ring.**

Proof: Since A is factorial, it is integrally closed, and the previous theorem applies. ■

DEFINITION: Let X be an affine variety, and $\hat{A}$ the integral closure of its ring of regular functions. Assume that $\hat{A}$ is a finitely generated ring. Then $\tilde{X} := \operatorname{Spec}(\hat{A})$ is called **normalization of X**.

REMARK: Using Noether's normalization lemma, **we shall prove that $\hat{A}$ is always finitely generated.**