Geometria Algébrica I

lecture 17: Discriminant

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Finite projection maps (reminder)

PROPOSITION: Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, z_i coordinates on $\mathbb{C}^n$, and $z_1, ..., z_k$ transcendence basis on $k(X)$. Then, for all $\lambda_1, ..., \lambda_k$ outside of the zero-set of a certain non-zero homogeneous polynomial, **the function $z_n \in \mathcal{O}_X$ is a root of a monic polynomial in the variables $z'_1, ..., z'_k$, where $z'_i := z_i + \lambda_i z_n$.**

Proof: Lecture 14. ■

Corollary 1: (Noether's normalization lemma, first version)
Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, z_i coordinates on $\mathbb{C}^n$, and $z_1, ..., z_k$ transcendence basis on $k(X)$. Then **there exists a linear coordinate change $z'_i := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$, such that the projection $\Pi_k: X \longrightarrow C^k$ to the first k arguments is a finite, dominant morphism.**

Proof: Previous proposition shows that the projection $P_n: X \longrightarrow \mathbb{C}^{n-1}$ is finite onto its image X_1 (after some linear adjustment). Using induction by n, we can assume that $P_k: X_1 \longrightarrow \mathbb{C}^k$ is also finite, hence the composition map is finite **(composition of finite morphisms is always finite, as we have seen).** ■

Noether's normalization lemma for non-irreducible varieties

The following version works for non-irreducible varieties.

PROPOSITION: Let $X \subset \mathbb{C}^n$ be an affine subvariety, and X_i its irreducible components. Denote by k the maximal transcendence degree for $k(X_i)$. Then **there exists a linear coordinate change $z_i' := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$, such that the projection $\Pi_k : X \longrightarrow C^k$ to the first k arguments is a finite.**

Proof. Step1: The natural projection map

$$\Psi : \mathcal{O}_X \longrightarrow \prod_{\mathfrak{m} \in \operatorname{Spec}(\mathcal{O}_X)} \mathcal{O}_X/\mathfrak{m}$$

is injective by Hilbert Nullstellensatz.

Step 2: The natural projection map $\Phi : \mathcal{O}_X \longrightarrow \bigoplus \mathcal{O}_{X_i}$ is injective, because Ψ factorizes through Φ. It is also finite, because $\mathcal{O}_{X_i}$ is finitely generated over $\mathcal{O}_X$. Clearly, $\coprod X_i = \operatorname{Spec}(\bigoplus \mathcal{O}_{X_i})$, where $\coprod$ denotes the disjoint union.

Step 3: Choose a coordinate projection $\Pi_k : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ which is finite on each X_i; such a projection exists by Corollary 1. The composition $\coprod X_i \longrightarrow X \xrightarrow{\Pi_k} \mathbb{C}^k$ is finite, hence $\bigoplus \mathcal{O}_{X_i}$ is a finitely generated $\mathcal{O}_{\mathbb{C}^k}$-module. Since $\mathcal{O}_{\mathbb{C}^k}$ is Noetherian, the submodile $\mathcal{O}_X \subset \bigoplus \mathcal{O}_{X_i}$ is also finitely generated. ■

Integral closure (reminder)

DEFINITION: Let $A \subset B$ be rings. The set of all elements in B which are integral over A is called **the integral closure of A in B**.

REMARK: The ring $\mathbb{C}[z_1, ..., z_n]$ **is factorial by Gauss lemma,** and therefore **integrally closed.**

THEOREM: Let A be an integrally closed Noetherian ring, $[K : k(A)]$ a finite extension of its field of fractions, and B the integral closure of A in K. **Then B is finitely generated as an A-module**.

Proof: Proven in Lecture 12. ■

EXAMPLE: Let $[K : \mathbb{C}(z_1, ..., z_n)]$ be a finite extension. **Then the integral closure of $C[z_1, ..., z_n]$ in X is finitely generated.**

REMARK: Since the normalization map is birational, it is **an isomorphism outside of a divisor**. Recall that **divisor** $S \subset M$ is an irreducible component of a subvariety given by an equation $f = 0$, where $f \in \mathcal{O}_M$ is a regular function which is not identically 0 on any irreducible component of M.

Normalization (reminder)

COROLLARY: Let X be an affine variety, and $\hat{A}$ the integral closure of its ring of regular functions. **Then $\hat{A}$ is finitely generated.**

Proof: The variety X admits a finite, dominant map to $\mathbb{C}^k$. Let A be the integral closure of $\mathbb{C}[z_1, ..., z_n]$ in $k(X)$; it is a finitely generated algebra by the previous theorem. Then A is an integrally closed ring containing $\mathcal{O}_X$ and with the same field of fractions. Since $A \supset \mathcal{O}_X \supset \mathbb{C}[z_1, ..., z_n]$, we obtain that A is finite over $\mathcal{O}_X$; this gives $A = \hat{A}$. ■

DEFINITION: Let X be an affine variety, and $\hat{A}$ the integral closure of its ring of regular functions. Then $\tilde{X} :=$ Spec$(\hat{A})$ is called **normalization of X**.

REMARK: The normalization map is finite and birational; **X is normal if for any finite, birational $\varphi: X' \longrightarrow X$, the map φ is an isomorphism**. Indeed, in this case $\mathcal{O}_{X'} \supset \mathcal{O}_X$ is finite with the same field of fractions.

COROLLARY: Normalization of X is a finite, birational morphism $X' \longrightarrow X$ **such that for any other finite, birational $\varphi: X'' \longrightarrow X'$, the map φ is an isomorphism**. In particular, **any birational, finite map $X' \longrightarrow X$ with X' normal is a normalization.** ■

Noether's normalization lemma (reminder)

THEOREM: (Noether's normalization lemma, second version) Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, and k the transcendence degree of X (number of elements in the transcendence basis of $[k(X):\mathbb{C}]$). Then there exists a variety $X_1 \subset \mathbb{C}^{k+1}$, given by a polynomial equation $P(t) = 0$, where $P(t)$ is a monic polynomial with coefficients in $\mathbb{C}[z_1, ..., z_k]$, such that **$X$ is isomorphic to the normalization of X_1.**

Proof. Step1: Let $X \subset \mathbb{C}^n$, with coordinates $z_1, ..., z_n$, and $z_1, ..., z_k$ a transcendence basis in $k(X)$. Using the same argument as used to prove the primitive element theorem, we find a primitive element $\tau = \sum_{i=k+1}^n \lambda_i z_i$

Step 2: Let Π_{k+1} be the projection to the coordinates $z_1, ..., z_k, \tau$, chosen in Step 1, and X_1 its image, that is, $X_1 = \operatorname{Spec}(B)$, where $B \subset \mathcal{O}_X$ is the subalgebra generated by $z_1, ..., z_k, \tau$. After an appropriate linear change of coordinates, we can assume that $\Pi_{k+1}: X \longrightarrow X_1$ is finite (Corollary 1) and birational (Step 1). Also, $\mathcal{O}_{X_1} = \mathbb{C}[z_1, ..., z_k, t]/(P)$ where $P(z_1, ..., z_k, t)$ is the monic polynomial constructed in Corollary 1.

Step 3: The projection $X \longrightarrow X_1$ is birational and finite, and X is normal. Therefore, X is normalization of X_1. ■

Multi-valued functions

DEFINITION: Define **a complex manifold** as a manifold equipped with a sheaf of functions which is locally isomorhic to a an open ball in $\mathbb{C}^n$ equipped with the sheaf of holomorphic functions. **This is the same as a manifold M with an atlas $\{U_i\}$, with each open subset $U_i \subset M$ identified with an open ball in $\mathbb{C}^n$, and complex analytic transition functions**.

REMARK: Define **complex variety** as a subvariety $Z \subset M$ in a complex manifold given by a collection of complex analytic equation.

DEFINITION: Multi-valued function on M is a closed, irreducible complex subvariety $Z \subset M \times \mathbb{C}$ such that the projection $Z \longrightarrow M$ is locally a diffeomorphism outside of a closed, nowhere dense subset in Z. The set Z is called **the graph of the multi-valued function**.

EXAMPLE: Logarithm is a multi-valued function on $\mathbb{C}$. Indeed, let Z be the graph of exponent $y = e^x$ in $\mathbb{C}^2$. The projection to x expresses all branches of logarithm $x = \log y$ as functions of y.

EXAMPLE: $y \longrightarrow \sqrt{y}$ is a multi-valued function. Indeed, the graph of $y = x^2$ projected to x gives both branches of $\sqrt{y}$.

Inverse/implicit function theorem (reminder)

THEOREM: ("Inverse function theorem")
Let $U, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a differentiable map. Suppose that the differential of f is everywhere invertible. **Then f is locally a diffeomorphism.**

DEFINITION: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a smooth function. A point $x \in U$ is a **critical point** of f if the differential $D_x f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is not surjective. **Critical value** is an image of a critical point. **Regular value** is a point of V which is not a critical value.

THEOREM: ("Implicit function theorem")
Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, $f: U \longrightarrow V$ a smooth function, and $y \in V$ a regular value of f. **Then $f^{-1}(y)$ is a smooth submanifold of U.**

Multi-valued functions and branched covers

THEOREM: Consider a subvariety $Z \subset \mathbb{C}^{n+1}$ given by a monic polynomial equation $P(t) = 0$, with $P(t) \in \mathcal{O}_{\mathbb{C}^n}[t]$. Assume that $P(t)$ is irreducible. **Then Z is a graph of a multi-valued function.** Moreover, Z is smooth, and the projection of Z to $\mathbb{C}^n$ (to the first n coordinates) is a diffeomorphism at (z,t) if and only if $P'(z) \neq 0$.

Proof. Step1: We shall represent points of $\mathbb{C}^{n+1}$ by pairs (z,t), with $z = (z_1, ..., z_n)$. Let $\pi: Z \longrightarrow \mathbb{C}^n$ be the standard projection along t. By the implicit function theorem, **$Z \subset \mathbb{C}^{n+1}$ is a smooth submanifold in a neighbourhood of any point $(z,t) \in Z$, with $z \in \mathbb{C}^n$ whenever the differential $dP: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ is surjective** (non-zero) at (z,t).

Step 2: This implies that Z is smooth outside of an algebraic subset of all $(z,t) \in \mathbb{C}^{n+1}$ such that $dP(z,t)(\xi,\tau) = 0$, for all $\xi \in \langle d/dz_1, ..., d/dz_n\rangle$, and $\tau \in \langle d/dt\rangle$. Let $P(z,t) = t^n + \sum_{i=0}^{n-1} t^i a_i(z)$. Then

$$dP(z,t) = nt^{n-1}dt + \sum_{i=0}^{n-1} t^i da_i(z) + \sum_{i=0}^{n-1} it^{i-1}a_i(z)dt.$$

For $|t| \gg 0$, the leading term $nt^{n-1}dt + t^{n-1}da_{n-1}(z)$ dominates the rest, and it is non-zero, because its dt component is non-zero.

Multi-valued functions and branched covers (2)

THEOREM: Consider a subvariety $Z \subset \mathbb{C}^{n+1}$ given by a monic polynomial equation $P(t) = 0$, with $P(t) \in \mathcal{O}_{\mathbb{C}^n}[t]$. Assume that $P(t)$ is irreducible. **Then Z is a graph of a multi-valued function.** Moreover, Z is smooth, and the projection of Z to $\mathbb{C}^n$ (to the first n coordinates) is a diffeomorphism at (z,t) if and only if $P'(z) \neq 0$.

Step 3: Let $z \in Z$ be a smooth point, and $(\xi, \tau) \in T_z Z$. Then $\pi : W \longrightarrow \mathbb{C}^n$ is invertible whenever W does not contain a vector $(0, \tau)$, equivalently, when $dP(z,t)(0,\tau) \neq 0$. **This is equivalent to $\frac{dP(z,t)}{dt} \neq 0$.**

Step 4: Let $P'(z,t) := \frac{dP(z,t)}{dt}$. To prove that Z defines a multi-valued function, it remains to show that $P'(z,t)$ is not identically zero on Z. Since $P(z,t)$ is irreducible, and $\mathcal{O}_{\mathbb{C}^{n+1}}$ is factorial, the ring $\frac{\mathcal{O}_{\mathbb{C}^{n+1}}}{(P)}$ has no zero divisors. Then Hilbert Nullstellensatz would imply that any function $f \in \mathcal{O}_{\mathbb{C}^{n+1}}$ which vanishes on Z is divisible by $P(z,t)$. Then $P'(z,t)$ does not vanish on Z, because it is polynomial of smaller degree. ∎

Symmetric polynomials

DEFINITION: **Symmetric polynomial** $P(z_1, ..., z_n) \in \mathbb{C}[z_1, ..., z_n]$ is a polynomial which is invariant with respect to the symmetric group Σ_n acting on $\mathbb{C}[z_1, ..., z_n]$ in a usual way.

DEFINITION: Consider the polynomial $P(z_1, ..., z_n, t) := \prod_{i=1}^n (t + z_i) = \sum e_i t^i$, with $e_i \in \mathbb{C}[z_1, ..., z_n]$. Then e_i are called **elementary symmetric polynomials** on $z_1, ..., z_n$.

THEOREM: **Every symmetric polynomial on $z_1, ..., z_n$ can be polynomially expressed through the elementary symmetric polynomials.**

Proof: Left as an exercise. ■

Discriminant of a polynomial

DEFINITION: Consider the symmetric polynomial $\prod_{i\neq j}(z_i - z_j)$. **Discriminant** of the polynomial $P(z_1, ..., z_n, t) := \prod_{i=1}^n (t - z_i)$ is $\prod_{i\neq j}(z_i - z_j)$ considered as a polynomial of its coefficients.

EXAMPLE: Discriminant of the quadratic polynomial $t^2 + bt + c$ is $b^2 - 4c$.

EXAMPLE: Discriminant of the cubic polynomial $t^3 + bt^2 + ct + d$ is $b^2c^2 - 4c^3 - 4b^3d - 27d^2 + 18bcd$.

CLAIM: A polynomial has no multiple roots if and only if its discriminant is non-zero. ■

Corollary 1: Let $P(t) \in k[t]$ be a polynomial over an algebraically closed field, and D its discriminant. **Then the derivative $P'(t)$ does not vanish on all roots of t if and only if $D \neq 0$.**

Proof: Let α be a root of P. Then $P(t) = P_\alpha(t)(t - \alpha)$, and $P'(t) = P'_\alpha(t)(t - \alpha) + P_\alpha(t)$. Therefore, $P'(\alpha) = 0$ if and only if $P_\alpha(t) = 0$. This is equivalent to $P(t)$ being divisible by $(t - \alpha)^2$. ■

Discriminant and ramified coverings

THEOREM: Consider a subvariety $Z \subset \mathbb{C}^{n+1}$ given by a monic polynomial equation $P(t) = 0$ of degree d, with $P(t) \in \mathbb{C}[z_1, ..., z_n][t]$, and let $\pi : Z \longrightarrow \mathbb{C}^n$ be the projection to the coordinates $z_1, ..., z_n$. Assume that $P(t)$ is irreducible. Denote by $D(z)$ the discriminant of $P(t)$, considered as a polynomial function on $(z_1, ..., z_n)$, and let $U \subset \mathbb{C}^n$ be the set of all $z \in \mathbb{C}^n$ such that $D(z) \neq 0$. **Then the intersection $Z \cap \pi^{-1}(U)$ is smooth and the projection $\pi : Z \cap \pi^{-1}(U) \longrightarrow U$ is a d-sheeted covering.**

Proof. Step1: By Corollary 1, for any $z \in U$, the polynomials $P(z,t)$ and $P'(z,t)$ have no common roots. Therefore, $dP(z,t) \neq 0$ on $Z \cap \pi^{-1}(U)$, and **the set $Z = \{(z,t) \mid P(z,t) = 0\}$ is smooth outside of zeros of $D(z)$.**

Step 2: Let $z \in Z \cap \pi^{-1}(U)$ and $(\xi, \tau) \in T_{(z,t)}Z$. Then $\pi : T_{(z,t)}Z \longrightarrow \mathbb{C}^n$ is invertible whenever $T_{(z,t)}$ does not contain a vector $(0, \tau)$, equivalently, when $dP(z,t)(0,\tau) \neq 0$. **This is equivalent to $\frac{dP(z,t)}{dt} \neq 0$.**

Step 3: The map $\pi : Z \cap \pi^{-1}(U) \longrightarrow U$ is locally a diffeomorphism, and each point has precisely d preimages. **Then it is a covering (prove it).** ■

Every algebraic variety is a ramified cover

Comparing this with the Noether normalization lemma, we obtain the following theorem.

COROLLARY: Let X be an algebraic variety. Then there exists a birational, finite map $X \longrightarrow Z$, a divizor $D \subset Z$, and a divisor $D_1 \subset \mathbb{C}^n$, **such that $Z \backslash D$ is a d-sheeted covering of $\mathbb{C}^n \backslash D_1$.** ■

COROLLARY: Every algebraic variety X over $\mathbb{C}$ has a smooth point. Moreover, **non-smooth points of X are contained in a proper algebraic subvariety of X.**

Proof: Indeed, **every birational map is an isomorphism outside of a divisor.** ■