Geometria Algébrica I

lecture 18: Dimension

Misha Verbitsky

IMPA, sala 232

October 22, 2018

Every algebraic variety is a ramified cover (reminder)

COROLLARY: Let X be an algebraic variety. Then there exists a birational, finite map $X \longrightarrow Z$, a divizor $D \subset Z$, and a divisor $D_1 \subset \mathbb{C}^n$, **such that $Z \backslash D$ is a d-sheeted covering of $\mathbb{C}^n \backslash D_1$.** ■

COROLLARY: Every algebraic variety X over $\mathbb{C}$ has a smooth point. Moreover, **non-smooth points of X are contained in a proper algebraic subvariety of X.**

Proof: Indeed, **every birational map is an isomorphism outside of a divisor.** ■

Finite, dominant maps are ramified covers

DEFINITION: A map $f: M_1 \longrightarrow M_2$ of smooth (or complex) manifolds is called **étale** if it is locally a diffeomorphism, that is, each point $x \in M_1$ has a naighbourhood $U \ni x$ such that $f: U \longrightarrow f(U)$ is a diffeomorphism.

CLAIM: Let M be an irreducible affine variety, and $\pi: M \longrightarrow \mathbb{C}^n$ the finite, dominant map constructed in Noether's normalization lemma. **Then π is étale outside of a proper algebraic subvariety $Z \subset M$.**

Proof. Step1: Using the primitive element theorem as in Lecture 16, we can represent the map $\pi: M \longrightarrow \mathbb{C}^n$ as a composition $M \xrightarrow{\psi} M_1 \xrightarrow{\pi_1} \mathbb{C}^n$, where ψ is birational, $\pi_1: M_1 \longrightarrow \mathbb{C}^n$ is a finite map, and $M_1 \subset \mathbb{C}^{n+1}$ is defined by a polynomial equation $P(t) = 0$, where $P(t) \in \mathcal{O}_{\mathbb{C}^n}[t]$ is a monic polynomial.

Step 2: As shown in Lecture 17, π_1 is a covering outside of the set of zeros of discriminant $D(P)$ of P. Let Z_1 be the set of points $x \in M$ such that $D(P)(\pi(x)) = 0$, and Z_2 be the divisor in M such that ψ defines an isomorphism of Z_2 to its image. Then $\pi: M \longrightarrow \mathbb{C}^n$ is étale in $M \backslash Z$, where $Z = Z_1 \cup Z_2$. ■

Transcendental dimension

REMARK: Let M be an irreducible affine variety, and $\pi: M \longrightarrow \mathbb{C}^n$ the finite, dominant map constructed in Noether's normalization lemma. By construction, n is equal to the transcendence degree of $[k(M):\mathbb{C}]$.

DEFINITION: Let M be an irreducible affine variety. **Dimension** of M is dimension of the smooth part of M, considered as a complex manifold. **Transcendental dimension** of M is the transcendence degree of $[k(M):\mathbb{C}]$.

PROPOSITION: **Dimension of M is equal to its transcendental dimension.**

Proof: Let $\pi: M \longrightarrow \mathbb{C}^n$ be the finite, dominant map. Then $k(M)$ is a finite extension of $\mathbb{C}[z_1, ..., z_n]$, hence n is equal to the transcendence degree of $[k(M):\mathbb{C}]$. On the other hand, π is étale outside of a proper algebraic subset, hence $\dim M = n$ as well. ■

Inverse/implicit function theorem (reminder)

THEOREM: ("Inverse function theorem")
Let $U, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a differentiable map. Suppose that the differential of f is everywhere invertible. **Then f is locally a diffeomorphism.**

DEFINITION: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a smooth function. A point $x \in U$ is a **critical point** of f if the differential $D_x f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is not surjective. **Critical value** is an image of a critical point. **Regular value** is a point of V which is not a critical value.

THEOREM: ("Implicit function theorem")
Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, $f: U \longrightarrow V$ a smooth function, and $y \in V$ a regular value of f. **Then $f^{-1}(y)$ is a smooth submanifold of U.**

Dimension of a divisor

DEFINITION: Let X be an affine variety, and $f \in \mathcal{O}_X$ a regular function which does not vanish on any of irreducible components of X. The zero set of f is called **a principal divisor** on X. Its irreducible components are called **divisors** on X.

Proposition 1: Let D be an irreducible divisor in $\mathbb{C}^n$. Then $\dim D = n-1$.

Proof. Step1: Let $P(z) \in \mathcal{O}_{\mathbb{C}^n} = \mathbb{C}[z_1, ..., z_n]$ be a polynomial. Then the irreducible components of the polynomial P correspond to irreducible components of the zero divizor of P. Therefore, **an irreducible divisor $D \subset \mathbb{C}^n$ is always obtained as the zero set of an irreducible polynomial $P(z)$.**

Step 2: For some i (say, $i = 1$), the derivative $Q := \frac{dP}{dz_1}$ is non-zero. Since $\deg Q < \deg P$, the polynomial Q does belong to the ideal (P) generated by P. By Hilbert Nullstellenzatz, any polynomial which vanishes in D belongs to (P). Therefore, Q is non-zero at some points of D. **In these points, the differential dP is surjective, hence D is smooth.**

Step 3: Let $z \in D$ be a point where $Q \neq 0$. The coordinate projection π to $(z_2, z_3, ..., z_n)$ is an isomorphism on $\ker dP\Big|_{T_zD}$, because $\frac{dP}{dz_1} \neq 0$, and $\ker d\pi = (t, 0, 0, ..., 0)$. **Therefore, π is a diffeomorphism in a neighbourhood of $z \in D$.** ■

Krull dimension

REMARK: Length of a chain $A_1 \subset A_2 \subset A_3 \subset ... \subset A_n$ is $n-1$, that is, the number of $\subset$ signs.

DEFINITION: Krull dimension of a ring A is the maximal possible length of a chain of prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 ... \subsetneq \mathfrak{p}_n \subsetneq A$

DEFINITION: Krull dimension of a variety X is the maximal possible length of a chain of non-empty, irreducible, distinct subvarieties $X_1 \subsetneq ... \subsetneq X_n$.

Today we are going to prove the following theorem

THEOREM: For any affine variety, **its dimension is equal to its Krull dimension.**

Local rings and Nakayama lemma

DEFINITION: A ring A is called **local** if it has only one maximal ideal.

DEFINITION: Let $\mathfrak{p} \subset A$ be a prime ideal, and $S \subset A$ its complement. **Localization of A in $\mathfrak{p}$** is $A[S^{-1}]$.

CLAIM: Localization $A_\mathfrak{p}$ of A in $\mathfrak{p}$ is local.

Proof: Any $x \in A \backslash \mathfrak{p}$ is invertible, hence $\mathfrak{p}$ **is a maximal ideal, containing all ideals in A.** ■

THEOREM: (Nakayama's lemma for local rings)
Let A be a Noetherian local ring, $\mathfrak{p}$ its maximal ideal, and M a finitely generated A-module. Then $M \supsetneq \mathfrak{p}M$.

Proof: For any non-trivial ideal $\mathfrak{a} \subset A$, **Nakayama lemma claims that $\mathfrak{a}M = M$ implies that $(1+a)M = 0$,** for some $a \in \mathfrak{a}$. For any $a \in \mathfrak{p}$, $1 + a$ is invertible, hence $M = 0$. ■

Finite ring extensions and prime ideals: going down

DEFINITION: Let $B \supset A$ be a ring, which is finitely generated as an A-module. In this case, we say that **B is finite extension of A.**

Lemma 1: Let $B \supset A$ be a extension of a ring A without zero divisors, and $\mathfrak{q} \subset B$ a non-zero prime ideal. **Them $\mathfrak{p} := \mathfrak{q} \cap A$ is nonzero.**

Proof: Consider the ring $A_\mathfrak{p} = A[S^{-1}]$ localized in the set S all $s \notin \mathfrak{p}$, and let $B_\mathfrak{p} := B[S^{-1}]$. Then $B_\mathfrak{p} \supset A_\mathfrak{p}$ is a finite $A_\mathfrak{p}$-module. If $\mathfrak{p} = 0$, $A_\mathfrak{p}$ is a field, and then $B_\mathfrak{p}$ is also a field as follows from the classification of semisimple Artinian algebras over a field. However, $B_\mathfrak{p}$ contains a non-trivial ideal $\mathfrak{q} \neq 0$, hence it cannot be a field. ■

Finite ring extensions and prime ideals: going up

Lemma 2: Let $B \supset A$ be a finite extension of a Noetherian ring A, and $\mathfrak{p} \subset A$ a prime ideal. **Then there exists finitely many prime ideals $\mathfrak{q} \subset B$ such that $\mathfrak{p} = \mathfrak{q} \cap A$.**

Proof. Step1: As above, consider the ring $A_\mathfrak{p} = A[S^{-1}]$ localized in the set S all $s \notin \mathfrak{p}$. The kernel of the natural map $A \longrightarrow A_\mathfrak{p}/\mathfrak{p}$ is $\mathfrak{p}$. Indeed, the map $A/\mathfrak{p} \longrightarrow A_\mathfrak{p}/\mathfrak{p}$ has no kernel because $\mathfrak{p}$ is prime, and the kernel of $A \longrightarrow A/\mathfrak{p}$ is $\mathfrak{p}$.

Step 2: Let $B_\mathfrak{p} := B[S^{-1}]$. By Nakayama's lemma, $B_\mathfrak{p} \neq \mathfrak{p} B_\mathfrak{p}$. Then $B_\mathfrak{p}/\mathfrak{p} = B_\mathfrak{p} \otimes_A A_\mathfrak{p}/\mathfrak{p}$ is a non-zero, finite-dimensional ring over the field $A_\mathfrak{p}/\mathfrak{p}$. Let $\tilde{\mathfrak{q}}$ be any prime ideal in $B_\mathfrak{p}/\mathfrak{p}$ (there are finitely many prime ideals by classification of Artinian algebras), and let $\mathfrak{q}$ be the preimage of $\tilde{\mathfrak{q}}$ inder the natural map $B \longrightarrow B_\mathfrak{p}/\mathfrak{p}$. Then $\mathfrak{q}$ is prime, and $\mathfrak{q} \cap A$ is mapped to 0 under the natural map $A \longrightarrow A_\mathfrak{p}/\mathfrak{p}$, hence $\mathfrak{q} \cap A = \mathfrak{p}$ (Step 1). ■

Cohen-Seidenberg theorems

THEOREM: (Cohen-Seidenberg theorem)
Let $B \supset A$ be a finite Noetherian ring over A, and $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 ... \subsetneq \mathfrak{q}_n \subsetneq B$ be a chain of prime ideals. Denote by $\mathfrak{p}_i$ the ideal $\mathfrak{p}_i \cap A \subset A$; it is clearly prime. Then

(i) $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 ... \subsetneq \mathfrak{p}_n \subsetneq A$ **(distinct prime ideals remain distinct)**

(ii) Any chain of prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 ... \subsetneq \mathfrak{p}_n \subsetneq A$ **is obtained this way.**

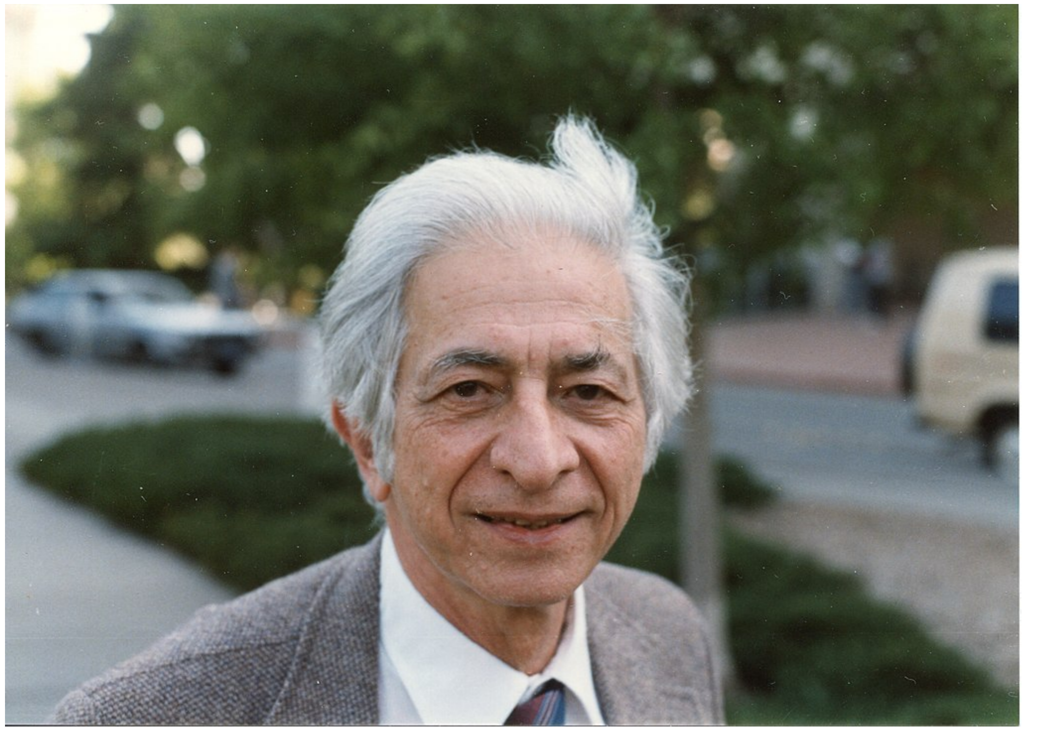
Proof of (i): Suppose that $\mathfrak{p}_i = \mathfrak{p}_{i-1}$. Replacing A by $A/\mathfrak{p}_{i-1}$ and B by $B/\mathfrak{q}_{i-1}$, we reduce the statement of (i) to Lemma 1.

Proof of (ii): Existence of $\mathfrak{q}_1$ follows from Lemma 2. Using induction, we may assume that $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 \subsetneq ... \subsetneq \mathfrak{q}_r$ is already chosen. To prove the induction step, we need to chose a prime ideal $\mathfrak{q}_{r+1}$ in $B/\mathfrak{q}_r$ such that $\mathfrak{q}_{r+1} \cap A/\mathfrak{q}_r = \mathfrak{p}_{r+1}$. This is again Lemma 2. ■

Irving S. Cohen and Abraham Seidenberg

Irving S. Cohen and Abraham Seidenberg, "Prime ideals and integral dependence", 1946, Bull. Amer. Math. Soc. 52 (4): 252-261

Irvin Sol Cohen
(1917-1955)



Abraham Seidenberg
(1916-1988)

Krull dimension is invariant under finite morphisms

COROLLARY: Let $X \longrightarrow Y$ be a finite, dominant morphism of irreducible affine varieties. **Then the Krull dimension of X is equal to Krull dimension of Y.**

Proof: Any chain of prime ideals in $\mathcal{O}_Y \subset \mathcal{O}_X$ can be lifted to $\mathcal{O}_X$ by Cohen-Seidenberg; any chain of distinct prime ideals in $\mathcal{O}_X$ intersected with $\mathcal{O}_Y$ gives a chain of distinct prime ideals in $\mathcal{O}_Y$, again by Cohen-Seidenberg. ■

The Krull dimension and the usual dimension

THEOREM: For any affine variety X, **its dimension** $\dim X$ **is equal to its Krull dimension** $\dim^k X$**.**

Proof. Step1: Using a finite, dominant map to $\mathbb{C}^n$ and the corollary above, we may assume that $X = \mathbb{C}^n$. Indeed, **a finite map to $\mathbb{C}^n$ does not change the Krull dimension and the usual dimension, as shown above.**

Step 2: Let D be an irreducible divisor in $\mathbb{C}^n$. The ideal I of polynomials vanishing in D is principal, $I = (P)$, where (P) is an irreducible polynomial. Therefore, there is no intermediate prime ideal $(P) \supsetneq \mathfrak{q} \supsetneq 0$. Conversely, for any prime ideal $I \subset \mathcal{O}_{\mathbb{C}^n}$, and any $P \in I$, at least one of the irreducible components P_i of P is contained in I, hence $I \supsetneq (P_i) \supsetneq 0$. Therefore, **in the maximal chain $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \ldots$ if prime ideals in $\mathcal{O}_{\mathbb{C}^n}$, the ideal $\mathfrak{p}_1$ is principal.**

Step 3: We obtained that $\dim^k \mathbb{C}^n = 1 + \dim^k D$. Using induction in $\dim X$, we may assume that $\dim Y = \dim^k Y$ for any affine variety of dimension $< n$. Then $\dim^k D = \dim D = n-1$ (Proposition 1), giving $\dim^k \mathbb{C}^n = \dim D + 1 = n$.

■