Geometria Algébrica I

lecture 19: Projective space and projective varieties

Misha Verbitsky

IMPA, sala 232

October 26, 2018

Complex projective space

DEFINITION: Let $V = \mathbb{C}^n$ be a complex vector space equipped with a Hermitian form h, and $U(n)$ the group of complex endomorphisms of V preserving h. This group is called **the complex isometry group**.

DEFINITION: Complex projective space $\mathbb{C}P^n$ is the space of 1-dimensional subspaces (lines) in $\mathbb{C}^{n+1}$.

REMARK: Since the group $U(n+1)$ of unitary matrices acts on lines in $\mathbb{C}^{n+1}$ transitively, $\mathbb{C}P^n$ **is a homogeneous space,** $\mathbb{C}P^n = \frac{U(n+1)}{U(1)\times U(n)}$, where $U(1) \times U(n)$ is a stabilizer of a line in $\mathbb{C}^{n+1}$.

EXAMPLE: $\mathbb{C}P^1$ is S^2.

Homogeneous and affine coordinates on $\mathbb{C}P^n$

DEFINITION: We identify $\mathbb{C}P^1$ with the set of $n+1$-tuples $x_0 : x_1 : ... : x_n$ defined up to equivalence $x_0 : x_1 : ... : x_n \sim \lambda x_0 : \lambda x_1 : ... : \lambda x_n$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordinates**. **Affine coordinates** in the chart $x_k \neq 0$ are are $\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k}$. The space $\mathbb{C}P^n$ is a union of $n+1$ affine charts identified with $\mathbb{C}^n$, with the complement to each chart identified with $\mathbb{C}P^{n-1}$.

CLAIM: Complex projective space is a complex manifold, with the atlas given by affine charts $\mathbb{A}_k = \left\{\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k}\right\}$, and the transition functions mapping the set

$$\mathbb{A}_k \cap \mathbb{A}_l = \left\{\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k} \;\middle|\; x_l \neq 0\right\}$$

to

$$\mathbb{A}_l \cap \mathbb{A}_k = \left\{\frac{x_0}{x_l} : \frac{x_1}{x_l} : ... : 1 : ... : \frac{x_n}{x_l} \;\middle|\; x_k \neq 0\right\}$$

as a multiplication of all terms by the scalar $\frac{x_k}{x_l}$.

Eigenvectors of commuting operators

LEMMA: Let $A, B \in \operatorname{End} V$ be commuting operators on a complex vector space V over an algebraically closed field. **Then A and B have a common eigenvector.**

Proof: Let V_α be an eigenspace of A with eigenvalue α. For each $v \in V_\alpha$, one has $AB(v) = BA(v) = B(\alpha v) = \alpha B(v)$. Therefore, $B(V_\alpha) \subset V_\alpha$. Now, any eigenvector of B in V_α is a common eigenvector. ■

LEMMA: Let $A_1, ..., A_n$ be a family of commuting operators on a complex vector space V. **Then A_1, ..., A_n have a common eigenvector.**

Proof: Using induction, we may assume that the operators $A_1, ..., A_k$ have a common eigenvector with eigenvalues $\alpha_1, ..., \alpha_k$. The corresponding eigenspace W is A_{k+1}-invariant, hence A_{k+1} has an eigenvector in W. This eigenvector is a common eigenvector for $A_1, ..., A_k, A_{k+1}$. Using induction in k, we obtain a common eigenvector for all A_i. ■

Representations of finite commutative groups

THEOREM: Let V be a complex representation of a commutative finite group G. **Then V is a direct sum of 1-dimensional G-representations.**

Proof: By the previous lemma, G has a common eigenvector, hence there exists a G-invariant 1-dimensional subspace $W \subset V$. **Choosing a G-invariant Hermitian structure, we obtain a G-invariant decomposition $V = W \oplus W^\perp$.** Using induction in $\dim V$, we may assume that the orthogonal complement $W^\perp$ is a a direct sum of 1-dimensional G-representations, hence the same is true for $V = W \oplus W^\perp$. ■

Algebraic groups

DEFINITION: Let $G \subset \mathbb{C}^n$ be an affine variety, equipped with a group structure, that is, the map $G \times G \xrightarrow{\mu} G$ of group multiplication and the map $G \xrightarrow{\iota} G$ of taking inverse satisfying the group axioms. We say that G is an **algebraic group** if the μ and ι are givem by algebraic morphisms, that is, expressed polynomially.

EXAMPLE: The space $\mathbb{C}^* := \mathbb{C}\backslash 0$ is given as an algebraic subset of $\mathbb{C}^2$ given by equation $xy = 1$. We define the group structure using the multiplication map $\mu((x_1, y_1), (x_2, y_2)) = (x_1 x_2, y_1 y_2)$ and the inverse $\iota(x, y) = (y, x)$. Clearly, **this group structure coincides with the usual multiplication on $\mathbb{C}^* \subset \mathbb{C}$.** Therefore, $(\mathbb{C}^*, \mu, \iota)$ is an algebraic group. This group is often denoted by $\mathbb{G}_m$, and called **the multiplicative group**.

Algebraic groups (2)

EXAMPLE: Let Det : $\mathbb{C}^{n^2}$ be the determinant map defined on the space $\mathbb{C}^{n^2} = \mathrm{Mat}(\mathbb{C}^n)$ of matrices in the usual way. The group $GL(n,\mathbb{C})$ is identified with the set $G \subset (\mathrm{Mat}(\mathbb{C}^n) \times \mathbb{C})$, defined by

$$G := \{(A,t) \in \mathrm{Mat}(\mathbb{C}^n) \times \mathbb{C} \quad | \quad \mathrm{Det}(A)t = 1\}.$$

The matrix multiplication gives multiplication on G: $\mu((A_1,t_1),(A_2,t_2)) = (A_1A_2, t_1t_2)$, and the standard formula for matrix inverse gives $\iota(A,t) = (t\,\mathrm{adj}(A), \det(A))$, where $\mathrm{adj}(A)$ is the adjugate matrix (one which is composed of (i,j)-minors of A). Clearly, the maps μ, ι **define the standard multiplicative structure on the group $G = GL(n,\mathbb{C})$ of invertible matrices.**

DEFINITION: A homomorphism of algebraic groups is a group homomorphism given by a morphism of the corresponding affine varieties.

DEFINITION: An algebraic representation of an algebraic group G is a morphism of algebraic groups $G \longrightarrow GL(n,\mathbb{C})$, that is, a representation expressed by polynomial maps from the group G to matrices.

Zariski topology (reminder)

DEFINITION: **Zariski topology** on an algebraic variety is a topology, where closed sets are algebraic subsets. **Zariski closure** of $Z \subset M$ is an intersection of all Zariski closed subsets containing Z.

DEFINITION: **Cofinite topology** is the topology on a set S such that the only closed subsets are S and finite sets.

EXERCISE: Prove that **Zariski topology on $\mathbb{C}$ coincides with the cofinite topology.**

CAUTION: **Zariski topology is non-Hausdorff.**

Zariski dense sets

DEFINITION: A subset $A \subset Z$ of an affine manifold is called **Zariski dense** if its closure in Zariski topology is Z.

EXERCISE: Let $A \subset \mathbb{C}$ be a subset. **Prove that A is Zariski dense if and only if it is infinite.**

CLAIM: Let $A \subset X$ be a Zariski dense subset, and $\varphi_1, \varphi_2 : X \longrightarrow Y$ be two algebraic maps. Suppose that $\varphi_1|_A = \varphi_2|_A$. **Then $\varphi_1 = \varphi_2$.**

Proof: The set $Z := \{x \in X \mid \varphi_1(x) = \varphi_2(x)\}$ is Zariski closed, because it is defined by polynomial equations. Since $A \subset Z$, **the set Z contains the Zariski closure of A.** ■

EXERCISE: Consider a differentiable group homomorphism $\varphi : S^1 \longrightarrow S^1$. **Prove that it is given by $t \longrightarrow t^d$, for some $d \in \mathbb{Z}$.**

Representations of algebraic groups

THEOREM: Any algebraic representation of $\mathbb{C}^*$ is a direct sum of 1-dimensional representations. Any 1-dimensional representation $\rho : \mathbb{C}^* \longrightarrow GL(1, \mathbb{C})$ is given by $\rho(z)(v) = z^k v$, for some $k \in \mathbb{Z}$.

Proof. Step1: Consider the subgroup $\mathbb{G}_{p^n} \subset \mathbb{C}^*$ of roots of unity of degree p^n. This is a commutative cyclic group, hence V can be represented as a sum of $\mathbb{G}_{p^n}$-invariant 1-dimensional representations, $V = \bigoplus V_i$. Consider a generator z of the cyclic group $\mathbb{G}_{p^n} \subset \mathbb{C}^*$. On each 1-dimensional representation V_i, $z \in \mathbb{G}_{p^n}$ acts by a scalar multiplication by z^{d_i}.

Step 2: Consider the numbers $d_i(n)$, obtained as functions of n in the sequence $\mathbb{G}_{p^n} \subset \mathbb{G}_{p^{n+1}} \subset \mathbb{G}_{p^{n+2}} \subset ....$ These numbers are compatible as follows: $d_i(n) = d_i(n+1) \mod p^n$. Passing to the closure of $\bigcup_n \mathbb{G}_{p^n}$, we obtain a smooth homomorphism from circle to circle represented by a polynomial map of degree $\leqslant N$. Then $d_i(n) \leqslant N$ for each n, and therefore the sequence $d_i(n)$ stabilizes. Denote by d_i its limit. Then $\rho(z) = z^{d_i}$ on V_i.

Step 3: Since $\bigcup_n \mathbb{G}_{p^n}$ is Zariski dense in $\mathbb{C}^*$, the $\bigcup_n \mathbb{G}_{p^n}$-invariant decomposition $V = \bigoplus V_i$ is actually $\mathbb{C}^*$-invariant. ■

Graded vector spaces

DEFINITION: Representation of $\mathbb{C}^*$ of finite type is a direct sum $V = \bigoplus V_i$ of algebraic representations of $\mathbb{C}^*$, with each V_i finite-dimensional, and with $\mathbb{C}^*$ acting on V_k by $\rho(z)(v) = z^k v$. **Morphism of representations of finite type** is a morphism of representations (that is, a linear map commuting with $\mathbb{C}^*$-action)

DEFINITION: A graded vector space is a vector space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$, represented as a direct sum of its **graded components** V_i. A graded vector space is called **of finite type** if all V^i are finitely dimensional. **Morphism** of graded vector spaces is a linear map preserving grading.

Claim 1: The category of graded spaces of finite type equivalent to the category of representations of $\mathbb{C}^*$ of finite type.

Proof: Let $\mathbb{C}^*$ act on $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$ as $\rho(z)(v) = z^k v$ on V^k. This gives a functor from graded vector spaces to representations. The inverse functor is provided by the eigenvalue decomposition for generic $z \in \mathbb{C}^*$. ■

Graded rings of finite type

DEFINITION: A graded ring is a ring A^*, $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with multiplication which satisfies $A^i \cdot A^j \subset A^{i+j}$ ("grading is multiplicative"). A graded ring is called **of finite type** if all A^i are finitely dimensional.

EXAMPLE: Polynomial ring $\mathbb{C}[V] = \bigoplus_i \operatorname{Sym}^i V$ is clearly graded.

DEFINITION: A ring with $\mathbb{C}^*$-action of finite type is a ring equipped with an action ρ of $\mathbb{C}^*$ of finite type, in such a way that the multiplication and addition is compatible with the $\mathbb{C}^*$-action: $\rho(z)(ab) = \rho(z)(a)\rho(z)(b)$, $\rho(z)(a+b) = \rho(z)(a) + \rho(z)(b)$.

PROPOSITION: The category of graded rings of finite type is equivalent to the category of rings with $\mathbb{C}^*$-action of finite type.

Proof: Let A^* be a graded ring, and let $\mathbb{C}^*$ act on $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$ as $\rho(z)(v) = z^k v$ on A^k. Then the compatibility of grading with multiplication implies $\rho(z)(ab) = \rho(z)(a)\rho(z)(b)$, giving a functor from graded rings to rings with $\mathbb{C}^*$-action. This construction is clearly invertible. ■

Projective varieties

DEFINITION: A projective variety is a subset of $\mathbb{C}P^n$ obtained as the set of solutions of a system homogeneous polynomial equations $P_1(z_1, ..., z_{n+1}) = P_2(z_1, ..., z_{n+1}) = ... = P_k(z_1, ..., z_{n+1}) = 0$.

DEFINITION: A graded ideal in a graded ring A^* is an ideal $I^* \subset A^*$ which is a direct sum of its graded components $I^* = \bigoplus I^k$, with $I^k \subset A^*$.

REMARK: Clearly, **a projective manifold is given by a graded ideal generated by $P_i(z_1, ..., z_{n+1})$.**

Theorem 1: Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ by homotheties, $\rho(t)(z) = tz$. Then **a $\mathbb{C}^*$-invariant subvariety $A \subset \mathbb{C}^{n+1}$ is given by a graded ideal, and conversely, each graded ideal defines a $\mathbb{C}^*$-invariant subvariety.**

Proof: The space of polynomials is graded by degree, and $\rho(t)$ acts on homogeneous polynomials of degree d as a multiplication by t^d. Since $\mathbb{C}^*$-invariant subspaces in $\mathbb{C}[z_1, ..., z_{n+1}]$ are the same as graded subspaces (Claim 1), $\mathbb{C}^*$-invariant subvarietes correspond to graded ideals. ■

Projective Nullstellensatz

REMARK: Let $I^* \subset R = \mathbb{C}[z_1, ..., z_{n+1}]$ be a graded ideal, and $\sqrt{I^*}$ its radical, that is, an ideal generated by all $x \in R$ such $x^n \in I^*$. **Then $\sqrt{I^*}$ is also graded.** Indeed, graded is the same as $\mathbb{C}^*$-invariant, and radical of an $\mathbb{C}^*$-invariant ideal is $\mathbb{C}^*$-invariant.

THEOREM: **$\mathbb{C}^*$-invariant radical ideals in $\mathbb{C}[z_1, ..., z_{n+1}]$ bijectively correspond to projective subvarieties in $\mathbb{C}P^n$.**

Proof: By Hilbert Nullstellensatz, the zero set $V(I)$ of a radical ideal $I^* \subset \mathbb{C}[z_1, ..., z_{n+1}]$ satisfies $\operatorname{Ann}(V(I)) = I$. By Theorem 1, this gives a bijective correspondence between non-empty $\mathbb{C}^*$-invariant subvarieties $\tilde{Z} \subset \mathbb{C}^{n+1}$ and $\mathbb{C}^*$-invariant radical ideals in $\mathbb{C}[z_1, ..., z_{n+1}]$. Since the set $\tilde{Z}$ is $\mathbb{C}^*$-invariant, with each its point $z \in \tilde{Z}$ it contains a line passing through z, hence it is uniquely determined by its projection to $\mathbb{C}P^n$. **We obtained a 1-to-1 correspondences between the following data:**

(subsets of $\mathbb{C}P^n$,obtained as solutions of a system of homogeneous equations)
$\Leftrightarrow$ (non-empty $\mathbb{C}^*$-invariant subvarieties in $\mathbb{C}^{n+1}$)
$\Leftrightarrow$ (radical graded ideals $I^* \subset \mathbb{C}[z_1, ..., z_{n+1}]$)

■