Geometria Algébrica I

lecture 20: Sheaves and algebraic varieties

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Sheaves

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below **is a more abstract version of the notion of "sheaf of functions"** defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Morphisms of sheaves

DEFINITION: Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

DEFINITION: **A sheaf isomorphism** is a homomorphism $\Psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi : \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$.

Some properties of Zariski topology

DEFINITION: **Base of topology** on a topological space M is a set $\{U_\alpha\}$ of open subsets such that any open subset of M can be obtained as a union of some of U_α, and intersections of any two U_α also belong to this family.

CLAIM: Let M be an affine variety. The base of Zariski topology on M can be given by all open subsets of form $M\backslash Z$, where Z is a principal divisor, that is, zero set of a function.

Proof: This is the same as to show that **any Zariski closed subset is an intersection of divisors.** ■

PROPOSITION: Any variety with Zariski topology is compact, that is, **any cover in Zariski topology has a finite subcover.**

Proof: Let $U_1 \subset U_2 \subset U_3 \subset ...$ be an increasing sequence of open subsets. To prove compactness, it would suffice to show that it stabilizes. However, the complements $M\backslash U_i$ give an decreasing sequence of Zariski closed subvarieties, that is, an increasing sequence of radical ideals, and such a sequence has to stabilize by Noetherianity. ■

Base of topology and sheaves

Proposition 1: Let $\mathfrak{S} = \{U_\alpha\}$ be a base of topology on a topological space M, and $\mathcal{F}(U_\alpha)$ a family of vector spaces, defined for each $U_\alpha \in \mathfrak{S}$. Assume that for each pair $U_\alpha \supset U_\beta$ from $\mathfrak{S}$, restriction maps are defined $\mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\beta)$, satisfying the sheaf axioms (associativity, gluing, vanishing). **Then there exists a unique sheaf $\mathcal{F}$ on M compatible with the sheaf data $\mathcal{F}(U_\alpha)$ for each $U_\alpha \in \mathfrak{S}$, and the restriction maps $\mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\beta)$.**

Proof: Let $U \subset M$ be an open set, $U = \bigcup_{i \in I} U_{\alpha_i}$, where $U_{\alpha_i} \in \mathfrak{S}$. **Define $\mathcal{F}(U)$ as the set of all families $f_i \in \mathcal{F}(U_{\alpha_i})$ which satisfy the gluing axiom** (this makes sense, because intersection of two elements of $\mathfrak{S}$ belongs to $\mathfrak{S}$). From the definition it is clear that $\mathcal{F}(U)$ is a presheaf; it is a sheaf because the gluing axioms for $\mathcal{F}(U_\alpha)$ immediately imply the gluing axioms for $\mathcal{F}(U)$. ■

Regularity is a local property of a function

REMARK: Note that **all subsets $M\backslash Z$, where Z is a principal divisor, are affine.**

Theorem 1: Let M be an affine variety, and $\{U_\alpha\}$ is a cover of M by affine varieties of form $U_\alpha = M\backslash Z_\alpha$, where Z_α is a principal divisor. Consider a function $f: M \longrightarrow \mathbb{C}$ which is regular on each U_α. **Then f is regular.**

Proof. Step1: Since M is compact, we can always assume that the set $\{U_\alpha\}$ is finite. Let Z_α be the zero divisor of $h_\alpha \in \mathcal{O}_M$. **Since $\bigcap Z_\alpha = \emptyset$, the functions h_α generate 1**, otherwise $\bigcap Z_\alpha$ would contain the common zeros of the ideal generated by h_α.

Step 2: By definition, the ring of regular functions on U_α is the localization $\mathcal{O}_M[h_\alpha^{-1}]$. Then $f(h_\alpha)^n$ is regular, for n sufficiently big (say, bigger than N). **Writing $1 = \sum_{\alpha=1^m} g_\alpha h_\alpha$ as in Step 1, we obtain that $f = f\left(\sum g_\alpha h_\alpha\right)^{Nm}$ is regular,** because it is a sum of monomials obtained as a product of f, regular functions, and h_α^N for at least one α. ■

Sheaf of regular functions

DEFINITION: Let $U \subset M$ be a Zariski open subset of an affine variety, obtained as a union $U = \bigcup U_\alpha$ of open affine subsets. We say that a function on U **is regular** if it is regular on U_α.

PROPOSITION: Regular functions constitute a sheaf.

Proof: Sheaf is constructed using Proposition 1. Gluing axiom follows from Theorem 1, the rest is clear. ■

DEFINITION: Algebraic variety (no longer "affine algebraic") is a compact topological space equipped with a ring of sheaves, which is locally isomorphic to an affine variety with its sheaf of regular functions and Zariski topology.

DEFINITION: Morphism of algebraic varieties is a map of algebraic varieties, continuous in Zariski topology, such that pullback of a regular function is regular.

REMARK: Let $f: M_1 \longrightarrow M_2$ be a morphism of affine varieties. Then a pullback of a regular function is regular. The coordinate functions $x_1, ..., x_n$ are regular, hence their pullbacks $f^*(x_i)$ are regular. The map f is given by polynomial functions $z \longrightarrow (f^*(x_i)(z), ..., f^*(x_n)(z))$, therefore **our two definitions of algebraic morphisms are compatible.**

Algebraic varieties: charts and atlases

As for the smooth manifolds, algebraic varieties can be defined in terms of charts and atlaces.

A chart on an algebraic variety is an open affine subset (a space with sheaf of functions which is isomorphic to an affine variety with the sheaf of regular functions). **An atlas** is a covering by affine charts $\{U_\alpha\}$, such that any intersection $U_\alpha \cap U_\beta$ is also a union of affine charts. **Gluing data** is transition functions $\varphi_{\alpha,\beta}$ from $U_\alpha \cap U_\beta \subset U_\alpha$ to $U_\alpha \cap U_\beta \subset U_\beta$. **Cocycle conditions** is $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ for any triple of charts $U_\alpha, U_\beta, U_\gamma$. Here the maps $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma}$ and $\varphi_{\alpha,\gamma}$ are considered as maps from the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ considered as a subset of U_α to $U_\alpha \cap U_\beta \cap U_\gamma$ considered as a subset of U_γ.

PROPOSITION: Let M be a topological space, and $\{U_\alpha\}$ a covering on M. Assume that each U_α is equipped with a sheaf of functions making it an affine variety, and the transition functions are algebraic and satisfy the cocycle condition. **Then M is equipped with a unique structure of an algebraic variety, compatible with this atlas and these transition functions**.

Proof: We recover the sheaf of regular functions on M using Proposition 1 to recover the sheaf of regular functions $\mathcal{O}_M$ on M. Then Theorem 1 implies that $\{U_\alpha\}$ is an affine cover. Then $(M, \mathcal{O}_M)$ is an algebraic variety. ■

Examples of algebraic varieties

EXAMPLE: Let $M \subset \mathbb{C}P^n$ be a projective variety. Then it is an algebraic variety in the sense of the definition above. Indeed, the homogeneous ideal I restricted to the affine set set $\mathbb{A}_k$ gives the ideal of $M \cap \mathbb{A}_k$ after setting $z_k = 1$. The subset $M \cap \mathbb{A}_k \cap \mathbb{A}_l$ is an affine subset given by $z_l \neq 0$, and the transition function maps

$$\mathbb{A}_k \cap \mathbb{A}_l = \left\{ \frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k} \;\middle|\; x_l \neq 0 \right\}$$

to

$$\mathbb{A}_l \cap \mathbb{A}_k = \left\{ \frac{x_0}{x_l} : \frac{x_1}{x_l} : ... : 1 : ... : \frac{x_n}{x_l} \;\middle|\; x_k \neq 0 \right\}$$

as a multiplication of all terms by $\frac{x_k}{x_l}$, hence it induces an isomorphism on regular functions. **The cocycle condition is apparent.**

EXAMPLE: Let $Z \subset M$ be a Zariski closed subset of an algebraic variety. **Then the complement $M \backslash Z$ is also an algebraic variety.** Indeed, locally Z is obtained as an intersection of divisors, and this gives a covering of $M \backslash Z$ by affine subvarieties.

REMARK: Note that $M \backslash Z$ is no longer affine, even if M is affine. Indeed, $\mathbb{C}^2 \backslash 0$ **is not affine.**

Sheaves of modules

REMARK: Let $A: \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \stackrel{\varphi_{U,U'}}{\longrightarrow} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: Locally free sheaf of modules over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

Coherent sheaves

DEFINITION: Let M be an algebraic variety. **Coherent sheaf** over M is a sheaf of finitely generated $\mathcal{O}_M$-modules.

DEFINITION: Let M be an algebraic variety. **A line bundle** over M is a locally free sheaf of rank 1 over the sheaf $\mathcal{O}_M$ of regular functions.

EXAMPLE: Trivial sheaf $\mathcal{O}_M$ is a line bundle. A line bundle is **trivial** if it is isomorphic to $\mathcal{O}_M$.

REMARK: The space of sections of a sheaf $\mathcal{F}$ over M is usually denoted **$H^0(\mathcal{F})$,** or $H^0(\mathcal{F})$ when it is clear.