Geometria Algébrica I

lecture 21: Veronese embedding

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IMPA, sala 232

November 5, 2018

Some properties of Zariski topology (reminder)

DEFINITION: Base of topology on a topological space M is a set $\{U_\alpha\}$ of open subsets such that any open subset of M can be obtained as a union of some of U_α, and intersections of any two U_α also belong to this family.

CLAIM: Let M be an affine variety. The base of Zariski topology on M can be given by all open subsets of form $M\backslash Z$, where Z is a principal divisor, that is, zero set of a function.

Proof: This is the same as to show that **any Zariski closed subset is an intersection of divisors.** ■

PROPOSITION: Any variety with Zariski topology is compact, that is, **any cover in Zariski topology has a finite subcover.**

Proof: Let $U_1 \subset U_2 \subset U_3 \subset ...$ be an increasing sequence of open subsets. To prove compactness, it would suffice to show that it stabilizes. However, the complements $M\backslash U_i$ give an decreasing sequence of Zariski closed subvarieties, that is, an increasing sequence of radical ideals, and such a sequence has to stabilize by Noetherianity. ■

Base of topology and sheaves

Proposition 1: Let $\mathfrak{S} = \{U_\alpha\}$ be a base of topology on a topological space M, and $\mathcal{F}(U_\alpha)$ a family of vector spaces, defined for each $U_\alpha \in \mathfrak{S}$. Assume that for each pair $U_\alpha \supset U_\beta$ from $\mathfrak{S}$, restriction maps are defined $\mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\beta)$, satisfying the sheaf axioms (associativity, gluing, vanishing) for such covers. **Then there exists a unique sheaf $\mathcal{F}$ on M compatible with the sheaf data $\mathcal{F}(U_\alpha)$ for each $U_\alpha \in \mathfrak{S}$, and the restriction maps $\mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\beta)$.**

Proof: Let $U \subset M$ be an open set, $U = \bigcup_{i \in I} U_{\alpha_i}$, where $U_{\alpha_i} \in \mathfrak{S}$. **Define $\mathcal{F}(U)$ as the set of all families $f_i \in \mathcal{F}(U_{\alpha_i})$ which satisfy the gluing axiom** (this makes sense, because intersection of two elements of $\mathfrak{S}$ belongs to $\mathfrak{S}$). From the definition it is clear that $\mathcal{F}(U)$ is a presheaf; it is a sheaf because the gluing axioms for $\mathcal{F}(U_\alpha)$ immediately imply the gluing axioms for $\mathcal{F}(U)$. ■

Sheaf of regular functions (reminder)

Theorem 1: Let M be an affine variety, and $\{U_\alpha\}$ is a cover of M by affine varieties of form $U_\alpha = M\backslash Z_\alpha$, where Z_α is a principal divisor. Consider a function $f: M \longrightarrow \mathbb{C}$ which is regular on each U_α. **Then f is regular.**

DEFINITION: Let $U \subset M$ be a Zariski open subset of an algebraic variety, obtained as a union $U = \bigcup U_\alpha$ of open affine subsets. We say that a function on U **is regular** if it is regular on U_α.

PROPOSITION: Regular functions constitute a sheaf.

Proof: Sheaf is constructed using Proposition 1. Gluing axiom follows from Theorem 1, the rest is clear. ■

DEFINITION: Algebraic variety (no longer "affine algebraic") is a topological space equipped with a ring of sheaves, which is locally isomorphic to an affine variety with its sheaf of regular functions and Zariski topology.

DEFINITION: Morphism of algebraic varieties is a map of algebraic varieties, continuous in Zariski topology, such that pullback of a regular function is regular.

Algebraic varieties: charts and atlases

As for the smooth manifolds, algebraic varieties can be defined in terms of charts and atlaces.

A chart on an algebraic variety is an open affine subset (a space with sheaf of functions which is isomorphic to an affine variety with the sheaf of regular functions). **An atlas** is a covering by affine charts $\{U_\alpha\}$, such that any intersection $U_\alpha \cap U_\beta$ is also a union of affine charts. **Gluing data** is transition functions $\varphi_{\alpha,\beta}$ from $U_\alpha \cap U_\beta \subset U_\alpha$ to $U_\alpha \cap U_\beta \subset U_\beta$. **Cocycle conditions** is $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ for any triple of charts $U_\alpha, U_\beta, U_\gamma$. Here the maps $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma}$ and $\varphi_{\alpha,\gamma}$ are considered as maps from the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ considered as a subset of U_α to $U_\alpha \cap U_\beta \cap U_\gamma$ considered as a subset of U_γ.

PROPOSITION: Let M be a topological space, and $\{U_\alpha\}$ a covering on M. Assume that each U_α is equipped with a sheaf of functions making it an affine variety, and the transition functions are algebraic and satisfy the cocycle condition. **Then M is equipped with a unique structure of an algebraic variety, compatible with this atlas and these transition functions**.

Proof: We recover the sheaf of regular functions on M using Proposition 1 to recover the sheaf of regular functions $\mathcal{O}_M$ on M. Then Theorem 1 implies that $\{U_\alpha\}$ is an affine cover. Then $(M, \mathcal{O}_M)$ is an algebraic variety. ■

Projective varieties (reminder)

DEFINITION: A projective variety is a subset of $\mathbb{C}P^n$ obtained as the set of solutions of a system homogeneous polynomial equations $P_1(z_1, ..., z_{n+1}) = P_2(z_1, ..., z_{n+1}) = ... = P_k(z_1, ..., z_{n+1}) = 0$.

DEFINITION: A graded ideal in a graded ring A^* is an ideal $I^* \subset A^*$ which is a direct sum of its graded components $I^* = \bigoplus I^k$, with $I^k \subset A^*$.

REMARK: Clearly, **a projective manifold is given by a graded ideal generated by $P_i(z_1, ..., z_{n+1})$.**

THEOREM: Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ by homotheties, $\rho(t)(z) = tz$. Then **a $\mathbb{C}^*$-invariant subvariety $A \subset \mathbb{C}^{n+1}$ is given by a graded ideal, and conversely, each graded ideal defines a $\mathbb{C}^*$-invariant subvariety.**

REMARK: Let $I^* \subset R = \mathbb{C}[z_1, ..., z_{n+1}]$ be a graded ideal, and $\sqrt{I^*}$ its radical, that is, an ideal generated by all $x \in R$ such $x^n \in I^*$. **Then $\sqrt{I^*}$ is also graded.** Indeed, graded is the same as $\mathbb{C}^*$-invariant, and radical of an $\mathbb{C}^*$-invariant ideal is $\mathbb{C}^*$-invariant.

THEOREM: $\mathbb{C}^*$-invariant radical ideals in $\mathbb{C}[z_1, ..., z_{n+1}]$ bijectively correspond to projective subvarieties in $\mathbb{C}P^n$.

Examples of algebraic varieties (reminder)

EXAMPLE: Let $M \subset \mathbb{C}P^n$ be a projective variety. Then it is an algebraic variety in the sense of the definition above. Indeed, the homogeneous ideal I restricted to the affine set set $\mathbb{A}_k$ gives the ideal of $M \cap \mathbb{A}_k$ after setting $z_k = 1$. The subset $M \cap \mathbb{A}_k \cap \mathbb{A}_l$ is an affine subset given by $z_l \neq 0$, and the transition function maps

$$\mathbb{A}_k \cap \mathbb{A}_l = \left\{ \frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k} \;\middle|\; x_l \neq 0 \right\}$$

to

$$\mathbb{A}_l \cap \mathbb{A}_k = \left\{ \frac{x_0}{x_l} : \frac{x_1}{x_l} : ... : 1 : ... : \frac{x_n}{x_l} \;\middle|\; x_k \neq 0 \right\}$$

as a multiplication of all terms by $\frac{x_k}{x_l}$, hence it induces an isomorphism on regular functions. **The cocycle condition is apparent.**

EXAMPLE: Let $Z \subset M$ be a Zariski closed subset of an algebraic variety. **Then the complement $M \backslash Z$ is also an algebraic variety.** Indeed, locally Z is obtained as an intersection of divisors, and this gives a covering of $M \backslash Z$ by affine subvarieties.

REMARK: Note that $M \backslash Z$ is no longer affine, even if M is affine. Indeed, $\mathbb{C}^2 \backslash 0$ **is not affine.**

Algebraic cones

DEFINITION: Let $M \subset \mathbb{C}P^n$ be a projective variety, defined by a graded ideal $I^* \subset \mathbb{C}[z_1, ..., z_{n+1}]$, and $C(M) \subset \mathbb{C}^{n+1}$ be the subset defined by the same ideal. Then $C(M)$ is called **the cone** or **the algebraic cone** of M.

REMARK: A subvariety $X \subset \mathbb{C}^{n+1}$ is a cone if and only if it is $\mathbb{C}^*$-invariant (here, as elsewhere, $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ by homotheties, $\rho(t)(v) = tv$). A $\mathbb{C}^*$-invariant subvariety determines M in a unique way.

DEFINITION: Projectivization of a homothety invariant subset $Z \subset \mathbb{C}^{n+1}$ is the set $Z_1 \subset \mathbb{C}P^n$ of all lines contained in Z. **In this case, $Z = C(Z_1)$.**

DEFINITION: The Graded ring of a projective variety is the ring of homogeneous functions on its cone. Using the notation defined above, it is a ring $\mathbb{C}[z_1, ..., z_{n+1}]/I^*$.

Families of homogeneous functions

DEFINITION: Let A^* be the graded ring of a projective variety, and $V \subset A^p$ a subspace. The set of **base points** of V the intersection of all zero divisors for all $f \in V$. The space $V \subset A^p$ is called **base point free** if it has no base point.

DEFINITION: Let $M \subset \mathbb{C}P^n$ be a projective manifold, and $L \subset A^d$ a base point free $(m+1)$-dimensional subspace, with basis $a_0, a_1, ..., a_m$. **Projective morphism associated with L** from M to $\mathbb{C}P^m$ is a map φ taking a point z with homogeneous coordinates $z_0 : z_1 : ... : z_n$ to $a_0(z) : a_1(z) : ... : a_m(z)$.

REMARK: If we replace $z_0 : z_1 : ... : z_n$ by an equivalent representation $\lambda z_0 : \lambda z_1 : ... : \lambda z_n$, for some number $\lambda \in \mathbb{C}^*$, the point $\varphi(z)$ is given by $\lambda^d a_0(z) : \lambda^d a_1(z) : ... : \lambda^d a_m(z)$, because all a_i are homogeneous of degree d. **Therefore, φ is a well defined morphism of algebraic varieties, $\varphi : M \longrightarrow \mathbb{C}P^m$.**

REMARK: It is possible to define projective morphisms in bigger generality, which I won't do today.

Homogeneous morphisms of algebraic cones

DEFINITION: Let $\rho(t)$ be the homothety action on $\mathbb{C}^n$, and $Z \subset \mathbb{C}^n$ a ρ-invariant subvariety (that is, a cone of a projective variety). We say that a morphism $\varphi: Z \longrightarrow \mathbb{C}^m$ **is homogeneous of degree d** if $\varphi(tv) = t^d\varphi(v)$.

REMARK: Let $X \subset \mathbb{C}P^n$ be a projective variety $\varphi: X \longrightarrow \mathbb{P}V$ be a projective morphism, defined by a base point free subspace $L \subset A^d$, where V is the dual space to L and $A^* = \bigoplus A^i$ the ring of functions on the corresponding cone $C(X) \subset \mathbb{C}^{n+1}$. Then φ defines a map of algebraic cones $C(X) \stackrel{C(\varphi)}{\longrightarrow} V$ associated with the ring homomorphism $\mathcal{O}_V = \operatorname{Sym}^*(L) \longrightarrow A^*$ mapping $l \in L$ to its image in A^*.

CLAIM: The map $C(X) \stackrel{C(\varphi)}{\longrightarrow} V$ is homogeneous of degree d.

Proof: $C(\varphi)$ takes a point $x = (x_0, ..., x_n) \in C(X)$ and maps it to $a_0(x), ..., a_m(x) \in \mathbb{C}^{m+1} = V$. This map is clearly homogeneous of degree d. ■

DEFINITION: Let $N := \dim \operatorname{Sym}^d(\mathbb{C}^{n+1})$. **Veronese embedding** is a map $\mathbb{C}P^n \longrightarrow \mathbb{C}P^{N-1}$ associated with the space $L = \operatorname{Sym}^d(\mathbb{C}^{n+1})$ of all degree d polynomials.

Veronese embedding

EXAMPLE: Veronese map $V: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ takes a point with homogeneous coordinates $x\colon y$ to $x^2\colon xy\colon y^2$. **Its image is a subvariety in $\mathbb{C}P^2$ given by a homogeneous equation $ac = b^2$.**

CLAIM: Veronese map $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ is an isomorphism from $\mathbb{C}P^1$ to a subvariety Z given by $ac = b^2$.

Proof. Step1: We cover Z by two charts, $U_a := \{(a\colon b\colon c) \in \mathbb{C}P^2 \ \mid\ a \neq 0\}$ and $U_c := \{(a\colon b\colon c) \in \mathbb{C}P^2 \ \mid\ c \neq 0\}$. Since $ac = b^2$, all points in Z with $b \neq 0$ belong to $U_a \cap U_c$, hence $U_a \cup U_c = Z$.

Step 2: In U_a, the map $\Psi: Z \longrightarrow \mathbb{C}P^1$ is defined by $a\colon b\colon c \mapsto 1 : \frac{b}{a}$. If $a\colon b\colon c = x^2\colon xy\colon y^2$, we have $\Psi(a\colon b\colon c) = 1\colon \frac{y}{x}$, hence it is inverse to V in the chart $1\colon z$ on $\mathbb{C}P^1$. In U_c, we define Ψ as $\Psi(a\colon b\colon c) = \frac{b}{c}\colon 1$. By the same reason, Ψ is inverse to V in the chart $z\colon 1$ in $\mathbb{C}P^1$. These maps agree on $U_a \cap U_c$, because in this set both a and c are invertible, giving

$$\left(\frac{b}{c}\colon 1\right) = (b\colon c) = (ab\colon ac) = (ab\colon b^2) = \left(1\colon \frac{b}{a}\right).$$

■