Geometria Algébrica I

lecture 22: Segre map and quadric hypersurfaces

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Decomposable tensors

DEFINITION: A rank of a linear map $V \longrightarrow W$ is dimension of its image.

DEFINITION: Let V, W be vector spaces. A tensor $\alpha \in V \otimes W$ is called **decomposable** if $\alpha = x \otimes y$, for some $x \in V, y \in W$.

CLAIM: $V \otimes W = \operatorname{Hom}(V^*, W)$ for any finitely-dimensional spaces V, W.

Proof: For any tensor $\alpha \in V \otimes W$ and $p \in V^*$, define $\zeta(p, v \otimes w) := \langle p, v\rangle w$. This map is linear on v, w, hence is extended to a linear map $V \otimes W \longrightarrow \operatorname{Hom}(V^*, W)$. This map is clearly injective. To see that it is an isomorphism, compare dimensions. ■

PROPOSITION: A tensor $\alpha \in V \otimes W$ is decomposable if and only if **the rank of the corresponding map $\kappa : V^* \longrightarrow W$ is $\leqslant 1$.**

Proof: Since $\zeta(p \otimes x \otimes y) = \langle p, x\rangle y$, the rank of κ is 1 for any decomposable α. Conversely, let $w \in W$ be a generator of the image of κ. Then $\alpha \in V \otimes \langle w\rangle$, and all elements in this space are decomposable. ■

Algebraic cones (reminder)

DEFINITION: Let $M \subset \mathbb{C}P^n$ be a projective variety, defined by a graded ideal $I^* \subset \mathbb{C}[z_1, ..., z_{n+1}]$, and $C(M) \subset \mathbb{C}^{n+1}$ be the subset defined by the same ideal. Then $C(M)$ is called **the cone** or **the algebraic cone** of M.

REMARK: A subvariety $X \subset \mathbb{C}^{n+1}$ is a cone if and only if it is $\mathbb{C}^*$-invariant (here, as elsewhere, $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ by homotheties, $\rho(t)(v) = tv$). A $\mathbb{C}^*$-invariant subvariety determines M in a unique way.

DEFINITION: **Projectivization** of a homothety invariant subset $Z \subset \mathbb{C}^{n+1}$ is the set $Z_1 \subset \mathbb{C}P^n$ of all lines contained in Z. **In this case, $Z = C(Z_1)$.**

DEFINITION: **The Graded ring** of a projective variety is the ring of homogeneous functions on its cone. Using the notation defined above, it is a ring $\mathbb{C}[z_1, ..., z_{n+1}]/I^*$.

Segre variety

COROLLARY: The set $Z \subset V \otimes W$ of decomposable tensors is an affine variety.

Proof: Let $v_1, ..., v_n$ be basis in V, and $w_1, ..., w_m$ basis in W. For any tensor $\alpha = \sum_{i,j} a_{ij} v_i \otimes w_j$, the rank of α considered as a map from V^* to W is equal to the rank of the matrix (a_{ij}). This matrix has rank 1 if and only if all 2×2 minors vanish. This is an algebraic condition. ■

DEFINITION: Let V, W be vector spaces. **Segre variety** is the projectivization of the set $Z \subset V \otimes W$ of decomposable tensors.

Product of affine varieties (reminder)

REMARK: Recall that **product** of objects X, Y in category $\mathcal{C}$ is an object $X \times Y$ such that $\mathfrak{Mor}(Z,X) \times \mathfrak{Mor}(Z,Y) = \mathfrak{Mor}(Z, X \times Y)$.

LEMMA: Let A, B be finitely-generated, reduced rings over $\mathbb{C}$, and $R := A \otimes_{\mathbb{C}} B$ their product. **Then R is reduced** (that is, has no nilpotents).

THEOREM: Let A, B be finitely generated rings without nilpotents, and $R := A \otimes_{\mathbb{C}} B$. **Then $\operatorname{Spec}(R) = \operatorname{Spec}(A) \times \operatorname{Spec}(B)$.** Moreover, **$\operatorname{Spec}(R)$ is the product of the varieties $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ in the category of affine varieties.**

Segre variety is a product

REMARK: Let X, Y be algebraic varieties, with affine covers $\{U_\alpha\}$ and $\{V_\beta\}$. The product $X \times Y$ is equipped with affine covers $\{U_\alpha \times V_\beta\}$, with all transition functions clearly regular, hence it is also an algebraic variety.

EXERCISE: Prove that **$X \times Y$ is a product of X and Y in the category of algebraic varieties.**

THEOREM: Let V, W be vector spaces, and Z the set of decomposable tensors in $V \otimes W$, and $S = \mathbb{P}Z$ the corresponding Segre variety. **Then S is the product of projective spaces $\mathbb{P}V$ and $\mathbb{P}W$.**

Proof. Step1: Let $\lambda : V \longrightarrow \mathbb{C}$ be a linear functional. Then λ defines a linear map $\Psi_\lambda : Z \longrightarrow W$ mapping $v \otimes w$ to $\lambda(v) \otimes w$. In the chart $U_\lambda \subset S$ given by $\Psi_\lambda(z) \neq 0$, this map defines a morphism of varieties $U_\lambda \longrightarrow \mathbb{P}W$, which is clearly independent from the choice of λ. Since $\bigcap_\lambda U_\lambda = \emptyset$, the natural projection $\pi_W : S \longrightarrow \mathbb{P}W$ is an algebraic morphism.

Step 2: The map $\pi_W \times \pi_V : S \longrightarrow \mathbb{P}V \times \mathbb{P}W$ is bijective and algebraic. To prove that it is an isomorphism, it remains to prove that the inverse map is also algebraic. However, the inverse map takes $v \in V, w \in W$ and maps them to $v \otimes w \in Z$; this map is polynomial. ■

Product of projective varieties

This gives a corollary

COROLLARY: A product of projective varieties is projective.

Proof. Step1: Let $X\subset \mathbb{C}P^n, Y\subset \mathbb{C}P^m$ be projective varieties. Then $X\times Y$ is a subvariety of the Segre variety $\mathbb{C}P^n\times\mathbb{C}P^m$ which is projective.

Step 2: It remains to show that an algebraic variety $X\subset\mathbb{C}P^n$ of a projective variety is projective. Then the cone $C(X)$ is an algebraic subvariety of $\mathbb{C}^{n+1}\backslash 0$. Locally in Zariski topology, $C(X)$ is defined by an ideal $I\subset \mathcal{O}_{U_i}$. Let $U_i:=\mathbb{C}^{n+1}\backslash D_{h_i}$, where h_i is a polynomial and D_{h_i} its zero set. Writing $1=\sum g_ih_i$ and replacing generators α_i of I by $\alpha_i\left(\sum_i g_ih_i\right)^N$ as in Lecture 20, we obtain that I can be generated by globally defined polynomials.

Step 3: Then $C(X)$ is an algebraic cone, and X is a projective variety, as proven in Lecture 19. ■

Projection with center in a point

DEFINITION: Let $H = \mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ be a hyperplane, associated to a vector subspace $V = \mathbb{C}^n \subset \mathbb{C}^{n+1} = W$, and $p \notin H$ a point in $\mathbb{C}P^n$. Given $x \in \mathbb{C}P^n \backslash \{p\}$, define **projection of x to H with center in p** as intersection $\pi(x) := V \cap \langle x, p\rangle$. **By construction, $\pi(x)$ is a 1-dimensional subspace in V, that is, a point in H.**

CLAIM: The projection map $\pi: \mathbb{C}P^n \backslash \{p\} \longrightarrow H$ is an algebraic morphism.

Proof: Assume that $V = \ker(\mu)$, where $\mu: W \longrightarrow \mathbb{C}$ is a linear functional. For any $x \in W\backslash\langle p\rangle$, one has $\pi(x) = \ker\mu \cap \langle x, p\rangle$. In affine coordinates this gives $\pi(x) = \mu(x)p - \mu(p)x$, which is clearly regular. ■

Veronese curve (reminder)

EXAMPLE: Veronese map $V: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ takes a point with homogeneous coordinates $x\colon y$ to $x^2\colon xy\colon y^2$. **Its image is a subvariety in $\mathbb{C}P^2$ given by a homogeneous equation $ac = b^2$.**

CLAIM: Veronese map $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ is an isomorphism from $\mathbb{C}P^1$ to a subvariety Z given by $ac = b^2$.

Proof. Step1: We cover Z by two charts, $U_a := \{(a\colon b\colon c) \in \mathbb{C}P^2 \mid a \neq 0\}$ and $U_c := \{(a\colon b\colon c) \in \mathbb{C}P^2 \mid c \neq 0\}$. Since $ac = b^2$, all points in Z with $b \neq 0$ belong to $U_a \cap U_c$, hence $U_a \cup U_c = Z$.

Step 2: In U_a, the map $\Psi: Z \longrightarrow \mathbb{C}P^1$ is defined by $a\colon b\colon c \mapsto 1 : \frac{b}{a}$. If $a\colon b\colon c = x^2\colon xy\colon y^2$, we have $\Psi(a\colon b\colon c) = 1\colon \frac{y}{x}$, hence it is inverse to V in the chart $1\colon z$ on $\mathbb{C}P^1$. In U_c, we define Ψ as $\Psi(a\colon b\colon c) = \frac{b}{c}\colon 1$. By the same reason, Ψ is inverse to V in the chart $z\colon 1$ in $\mathbb{C}P^1$. These maps agree on $U_a \cap U_c$, because in this set both a and c are invertible, giving

$$\left(\frac{b}{c}\colon 1\right) = (b\colon c) = (ab\colon ac) = (ab\colon b^2) = \left(1\colon \frac{b}{a}\right).$$

■

Quadric

DEFINITION: Let g be a non-degenerate bilinear symmetric form on $V = \mathbb{C}^{n+1}$. A **(non-degenerate) quadric** is a subset of $\mathbb{P}V$ given by an equation $g(x, x) = 0$.

CLAIM: All quadrics are isomorphic.

Proof: Indeed, all non-degenerate bilinear symmetric forms over $\mathbb{C}$ are related by a linear transform (use an orthonormal basis). ■

CLAIM: A 0-dimensional quadric is 2 points in $\mathbb{C}P^1$. ■

CLAIM: A 1-dimensional quadric Q_1 is isomorphic to $\mathbb{CP}^1$.

Proof: Indeed, Q_1 can be given by an equation $xy - z^2 = 0$, which is an equation for the Veronese curve. ■

Orthogonal group acts transitively on quadrics

CLAIM: The group $O(V)$ of orthogonal linear automorphisms of V **acts transitive on any non-degenerate quadric Q.**

Proof: Let $v \in V$ be a vector such that $g(v,v) = 0$, and w a vector such that $g(v,w) \neq 0$. Let $w_1 := \mu v + w$, where $\mu = -2\frac{g(w,w)}{g(v,w)}$. This vector satisfies $g(w_1, w_1) = g(w,w) + 2\mu g(v,w) = 0$ and $g(v, w_1) \neq 0$. Replacing w by $\frac{w_1}{g(v,w_1)}$, we may assume that $g(v,w) = 1$ and $g(w,w) = 0$. Denote by W the orthogonal complement to $V_0 := \langle v, w\rangle$, and choose an orthonormal basis $z_1, ..., z_n$ in W. The matrix of g in the basis $(v, w, z_1, ..., z_n)$ is written as

$$\begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & & & & \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$$

If v' is another non-zero vector with $g(v', v') = 0$, we find another basis $(v', w', z'_1, ..., z'_n)$ where g has the same matrix. **Then the linear map A putting v to v', w to w' and z_i to z'_i is orthogonal.** ■

COROLLARY: All (non-degenerate) quadrics are smooth.

Proof: Some points on a quadric are smooth (Lecture 17). Since the group $O(V)$ acts on quadric transitively, all points are equivalent. ■

All quadric are rational

DEFINITION: An algebraic variety over $\mathbb{C}$ is called **rational** if it is birational to $\mathbb{C}P^n$

PROPOSITION: All quadrics are rational.

Proof. Step1: Let $Q \subset \mathbb{C}P^n = \mathbb{P}V$ be a quadric defined by a quadratic form h, and $z \in Q$ a point. Consider the projection map $\xi \mathbb{P}V\backslash z \longrightarrow \mathbb{C}P^{n-1} = \mathbb{P}V_1$ with center in z. For a point $v \in Q$ distinct from z, denote by l_v the projective line $\mathbb{C}P^1$, associated with a 2-dimensional subspace $\langle z, v\rangle \subset V$ connecting v and z. A non-zero quadratic equation cannot have more than two solutions on $\mathbb{C}P^1$, hence the projection $\xi: Q\backslash z \longrightarrow \mathbb{C}P^{n-1}$ is genericaly 1-to-1.

Step 2: For any $x \in \mathbb{P}V_1$, the quadratic polynomial h restricted to $\langle z, x\rangle$ is divisible by a linear form λ which vanishes in z. This gives a linear form $\frac{h}{\lambda}$ on $\langle z, x\rangle$. Unless h vanishes on $\langle z, x\rangle$, the form $\frac{h}{\lambda}$ is non-zero, and gives a point in $Q \cap \mathbb{P}\langle z, x\rangle$. We obtained an inverse map to ξ. ■

Quadrics in $\mathbb{C}P^3$

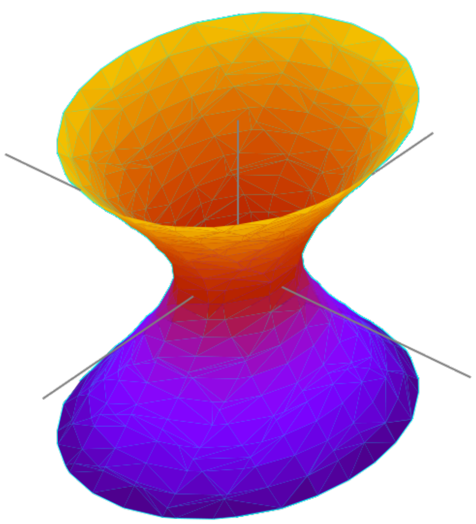
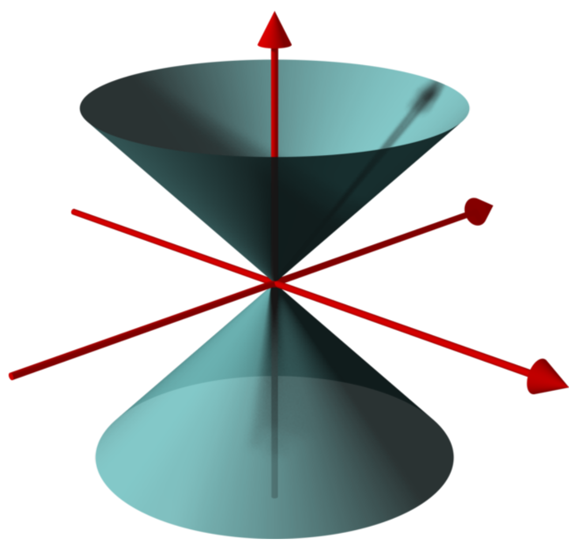
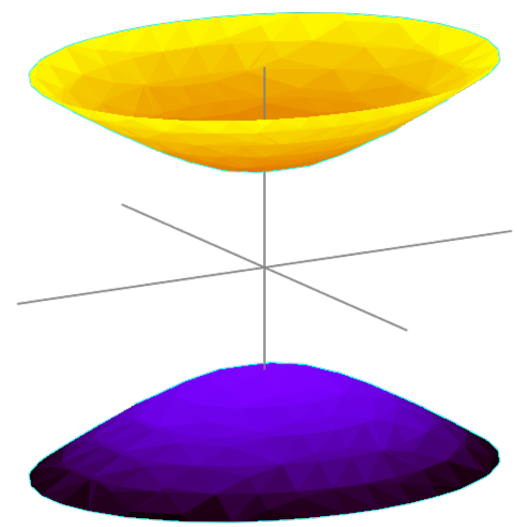
THEOREM: A non-degenerate quadric in $\mathbb{C}P^3$ **is isomorphic to the image of Segre embedding $\mathbb{C}P^1 \times \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^3$.**

Proof: Let V_1, V_2 be 2-dimensional complex spaces, and $V = V_1 \otimes V_2$. The tensor $\alpha \in V_1 \otimes V_2$ is decomposable if and only if its matrix (a_{ij}) is denegerate, which happens when $\det(a_{ij}) = a_{11}a_{22} - a_{12}a_{21} = 0$. However, $a_{11}a_{22} - a_{12}a_{21}$ is a non-degenerate quadratic form on V. ■

Affine quadrics in $\mathbb{R}^3$

DEFINITION: Let Q be a quadratic form on V and λ a linear form. Then the set $S := \{v \in V \ | \ Q(v) + \lambda(v) + c = 0\}$ is called **affine quadric**.

DEFINITION: Let Q be a quadratic form on $\mathbb{R}^3$ of signature $(2,1)$, and $c \neq 0$. The affine quadric $S := \{v \in V \ | \ Q(v) = c\}$ is called **hyperboloid**. When $c > 0$, it is called **hyperbolic**, or **one-sheeted hyperboloid**, or **ruled hyperboloid** and when $c < 0$, it is **elliptic**, or **two-sheeted hyperboloid**

Quadratic forms on $\mathbb{R}^2$

DEFINITION: Let q be a quadratic form on a vector space V. A vector $v \in V$ is called **isotropic** if $q(v) = 0$.

Proposition 1: Let $Q(xe_1 + ye_2) = ax^2 + by^2 + 2cxy$ be a non-degenerate quadratic form on $\mathbb{R}^2$. **Then the set $\{v \in \mathbb{R}^2 \ \mid\ Q(v) = 0\}$ if isotropic vectors is either a union of two lines intersecting in 0, or $\{0\}$ depending on signature.**

Proof: If Q is positive definite or negative definite, it is $\{0\}$. If the signature is $(1,1)$, let $u, v \in \mathbb{R}^2$ be the basis such that the corresponding bilinear symmetric form satisfies $q(u,u) = 1, q(v,v) = -1, q(u,v) = 0$. Then the vectors $w_+ := \frac{u+v}{2}$ and $w_- := \frac{u-v}{2}$ are isotropic and satisfy $q(w_+, w_-) = 1$. No linear combination of form $aw_+ + bw_-$, with $a, b \neq 0$ can be isotropic, because $q(aw_+ + bw_-) = a^2 q(w_+, w_+) + b^2 q(w_-, w_-) + 2abq(w_-, w_+) = 2ab$. ■

Ruled hyperboloid

PROPOSITION: Let Q be a quadratic form on $\mathbb{R}^3$ of signature $(2,1)$, $c > 0$. and $S := \{v \in V \ | \ Q(v) = c\}$ the corresponding hyperboloid. **Then for any tangent plane W, the intersection $W \cap S$ is union of two lines.**

Proof: Let s be the tangent point, and choose an affine coordinate system such that $s = 0$. Then the tangent plane W is linear, and $Q(v) - c$ is a quadratic form on W of signature (1,1) (Remark 1). **Then $W \cap S$ is a union of two lines by Proposition 1.** ■

DEFINITION: A 2-dimensional surface $S \subset \mathbb{R}^3$ is called **ruled** if each point of S is contained on a line $l \subset S$.

Quadrics of rotation

DEFINITION: Fix a positive definite form g on $\mathbb{R}^3$, and let Q be a non-degenerate quadratic form on $\mathbb{R}^3$. In appropriate orthonormal coordinates, Q can be written as

$$Q(x,y,z) = \pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2}. \quad (*)$$

The coordinate axes of this coordinate system are called **axes of the quadric** S. **They are defined uniquely up to an automorphism of $\mathbb{R}^3$ preserving g and Q.**

DEFINITION: We say that a quadric $S := \{v \in V \quad | \quad Q(v) = c\}$ **has rotational symmetry** if it is preserved by an isometric rotation of $\mathbb{R}^3$.

CLAIM: A quadric has rotational symmetry if and only if two of the coefficients in (*) are equal.

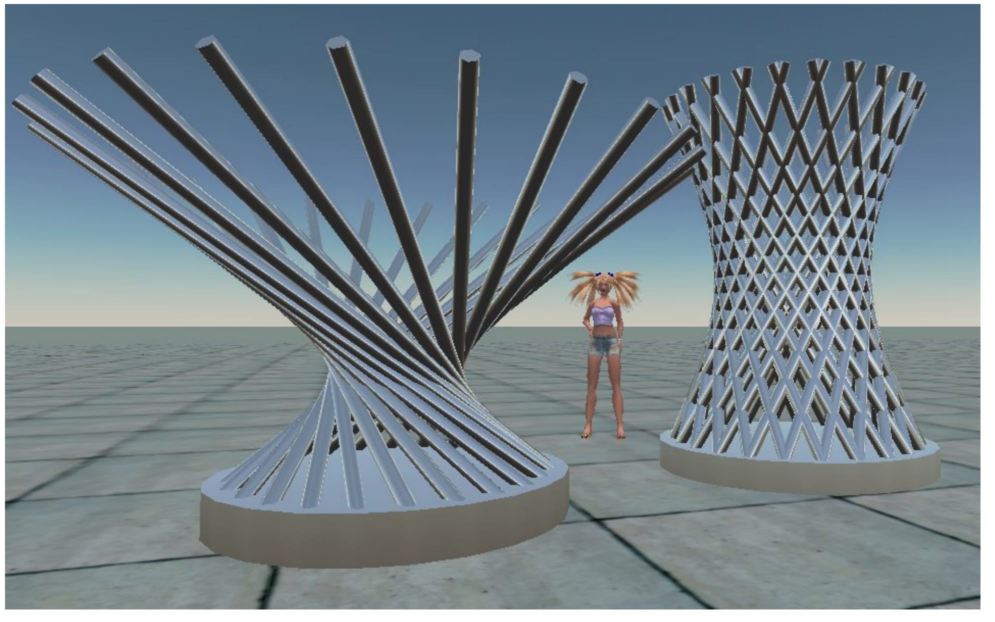
Proof: Clearly, if (say) two of the coefficients in (*) are equal and $Q(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} \pm \frac{z^2}{c^2}$, then rotation around an axis $x = y = 0$ preserves Q.

Conversely, when all coefficients in (*) are different, the orthonormal basis (x,y,z) is determined uniquely up to a sign, and any isometry preserving Q also preserves these three axes. ■

Hyperboloid of rotation

PROPOSITION: Let $l_1, l_2 \subset \mathbb{R}^3$ be two non-perpendicular skew lines (that is, likes which are not parallel and don't intersect), and S a surface of rotation obtained by rotating l_1 around l_2. **Then S is a hyperboloid of rotation.** Conversely, **any ruled hyperboloid of rotation is obtained this way**.

Proof: Let S be a hyperboloid of rotation. Then S contains a line. Since it is rotationally symmetric, it can be obtained by rotating this line around the central axis.



Conversely, any two non-perpendicular skew lines can be related by affine transform commuting with rotation around the first line **(prove it)**, and an affine transform maps quadrics to quadrics. ■