Geometria Algébrica I

lecture 23: Main theorem of elimination theory

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Subvarieties in affine varieties

LEMMA: Let X be an affine variety, and $Z \subset X$ an algebraic subvariety. **Then $Z \subset X$ is an affine subvariety.**

Proof. Step1: Clearly, it would suffice to prove the lemma when X is irreducible. Indeed, a union of affine subvarieties is affine.

Step 2: Let $\{U_{h_i}\}$ be an affine cover of X, with $U_{h_i} = X \backslash D_{h_i}$, where D_{h_i} is the zero divisor of a function $h_i \in \mathcal{O}_X$. Then $U_{h_i} = \operatorname{Spec}(\mathcal{O}_X[h_i^{-1}])$. Dropping some open sets if necessary, we may always assume that $Z \cap U_{h_i}$ is non-empty. Locally in each chart U_{h_i}, the variety Z is defined by an ideal $I_{h_i} \in \mathcal{O}_X[h_i^{-1}]$.

Step 3: Let $f_1, ..., f_n$ be the generators of I_{h_i}. Then $h_i^N f_1, ..., h_i^N f_n$ are regular functions which generate I_{h_i}. These functions vanish on Z, because it is irreducible, and they vanish in an open non-empty subset. Therefore, the collection of $h_i^N f_1, ..., h_i^N f_n$, taken for all h_i, is the ideal defining Z. ■

The same argument also proves the following assertion.

Lemma 1: Let X be an affine variety, $X_1 \subset X$ its affine subvariety, and $Z \subset X \backslash X_1$ an algebraic subvariety. **Then the Zariski closure $\overline{Z}$ of Z is an affine subvariety of X which satisfies $\overline{Z} \cap (X \backslash X_1) = Z$.** ■

Subvarieties in $\mathbb{C}P^n \times X$

Further on, we shall need the following proposition. It is not immediately clear, because **"algebraic subvariety" is defined locally in Zariski topology,** and the statement is global.

PROPOSITION: Let X be an affine variety, $Z \subset \mathbb{C}P^n \times X$ be an algebraic subvariety. Denote by $z_0 : ... : z_n$ the homogeneous coordinates on $\mathbb{C}P^n$. Then there exists a graded ideal $I^* \subset \mathcal{O}_X[z_0, ..., z_n]$ such that **X is the set of common zeros of I^*.**

Proof. Step1: The natural projection $(\mathbb{C}^{n+1} \backslash 0) \times X \xrightarrow{\Psi} \mathbb{C}P^n \times X$ is clearly algebraic. Therefore, $\Psi^{-1}(Z)$ is an algebraic subvariety of $(\mathbb{C}^{n+1} \backslash 0) \times X$, invariant under the natural $\mathbb{C}^*$-action.

Step 2: Let $Z_1 \subset \mathbb{C}^{n+1} \times X$ be the Zariski closure of $\Psi^{-1}(Z)$ in $\mathbb{C}^{n+1} \times X$. Lemma 1 implies that Z_1 is an affine subvariety of $\mathbb{C}^{n+1} \times X$.

Step 3: Let I be the ideal of Z_1. Since I is $\mathbb{C}^*$-invariant with respect to the standard $\mathbb{C}^*$-action on A^*, it is graded. ■

Empty projective varieties

Claim 1: Let $I^* \subset \mathbb{C}[z_0, ..., z_n]$ be a graded ideal, and $Z \subset \mathbb{C}P^n$ the set of common zeros of I^*. **The set Z is empty if and only if for some N, the graded component I^d coincides with the set $\mathbb{C}[z_0, ..., z_n]^d$ of all polynomials of degree d.**

Proof: If $I^d = \mathbb{C}[z_0, ..., z_n]^d$, the set Z is clearly empty. Conversely, if Z is empty, this means that its cone $C(Z)$ does not contain any lines, hence the radical $R(I^*) := \{u \in \mathbb{C}[z_0, ..., z_n] \ | \ u^n \in I^*\}$ of I^* is the ideal of 0. This implies that for some $N \gg 0$, one has $z_i^N \in I^*$. However, any monomial of degree $\geqslant (n+1)N$ is divisible by z_i^N for some i, hence $I^{(n+1)N} = \mathbb{C}[z_0, ..., z_n]^{(n+1)N}$.
■

Specialization

DEFINITION: A family of algebraic varieties, parametrized by X is an algebraic variety Y equipped with a morphism $\Phi : Y \longrightarrow X$. For each $x \in X$, the preimage $\Phi^{-1}(x)$ is called **the specialization** of a family Y to the point $x \in X$.

REMARK: Assume that X is affine. Let $\mathfrak{m}$ be the maximal ideal of x, and $p: \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathfrak{m} = \mathbb{C}$ be the quotient map. As shown in Lecture 8, **the ring of regular functions on the fiber $Y_x = \Phi^{-1}(x)$ is the quotient of $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}$ by the nilradical.**

Image of projective varieties

The main result of today's lecture is the following theorem.

THEOREM: ("Fundamental theorem of elimination theory") Let $Z \subset \mathbb{C}P^n \times X$ be an algebraic subvariety, and $\pi: \mathbb{C}P^n \times X \longrightarrow X$ the projection. **Then $\pi(Z)$ is algebraic in X.**

Proof: Since the statement is local in X, we may assume that X is affine. **Then Z is defined by a graded ideal $I^* \subset R := \mathcal{O}_X[z_0, ..., z_n]$ (Lemma 1).** Let $w_1, ..., w_r$ be generators of the ideal I^*. We can consider $w_1, ..., w_r$ as regular functions on X with values in the space $\mathbb{C}[z_0, ..., z_n]^{\leqslant d}$ of homogeneous polynomials of degree $\leqslant d$. A point $x \in X$ belongs to $\pi(z)$ if and only if $w_1(x), ..., w_r(x)$ do not generate $\mathbb{C}[z_0, ..., z_n]^d$ for any $d > 0$ (Claim 1). We reduced "Fundamental Theorem" to the following proposition.

PROPOSITION: Let X be an affine variety, and $w_1, ..., w_r: X \longrightarrow \mathbb{C}[z_0, ..., z_n]^{\leqslant d}$ regular functions with values in the space $\mathbb{C}[z_0, ..., z_n]^{\leqslant d}$. Denote by $V \subset X$ the set of all $x \in X$ such that the ideal generated by $w_1, ..., w_r$ does not contain $\mathbb{C}[z_0, ..., z_n]^d$ for any $d > 0$. **Then V is an algebraic subvariety in X.**

We shall prove it later today.

Macaulay matrices

DEFINITION: Let $W \subset \mathbb{C}[z_0, ..., z_n]^d$ be a subspace in the space of homogeneous polynomials generated by $w_1, ..., w_r$, and $M_W^s : \mathbb{C}[z_0, ..., z_n]^{s-d} \otimes W \longrightarrow \mathbb{C}[z_0, ..., z_n]^s$ be the multiplication map. Then M_W^s is called **the Macaulay matrix**. The rows of this matrix are enumerated by the pairs (w_i, θ), where θ is a monomial of degree $n - d$, and the columns by monomials ξ of degree n. If we write $w_i\theta = \sum \alpha_j^{i,\theta} \xi_j$, the entry of M_W^s corresponding to $((w_i, \theta), \xi)$ is given by $\alpha_j^{i,\theta}$.

CLAIM: Let $I^* \subset \mathbb{C}[z_0, ..., z_n]$ be a graded ideal generated by $W \subset \mathbb{C}[z_0, ..., z_n]^d$. **Then I defines a non-empty projective variety if and only if the Macaulay matrix is not surjective for all $s > 0$.**

Proof: By Claim 1, I^* defines a non-empty projective variety if and only if $I^s \neq \mathbb{C}[z_0, ..., z_n]^s$ for all $s > 0$. However, $I^s = \operatorname{im} M_W^s$. ■

Macaulay matrices and elimination

PROPOSITION: Let X be an affine variety, and $w_1, ..., w_r : X \longrightarrow \mathbb{C}[z_0, ..., z_n]^{\leqslant d}$ regular functions with values in the space $\mathbb{C}[z_0, ..., z_n]^{\leqslant d}$. Denote by $V \subset X$ the set of all $x \in X$ such that the ideal generated by $w_1, ..., w_r$ does not contain $\mathbb{C}[z_0, ..., z_n]^d$ for any $d > 0$. **Then V is an algebraic subvariety in X.**

Proof. Step1: Let I^* be the ideal generated by $\{w_i\}$. Replacing $\{w_i\}$ by generators of I^d, **we may assume that all w_i have the same degree.** Denote by W the space generated by w_i.

Step 2: Let V_s be the set of all $x \in X$ such that the Macaulay matrix M_W^s : $\mathbb{C}[z_0, ..., z_n]^{s-d} \otimes W \longrightarrow \mathbb{C}[z_0, ..., z_n]^s$ is non-surjective. Let $m := \dim \mathbb{C}[z_0, ..., z_n]^s$. **Then V_s is the set of all $x \in X$ such that all $m \times m$-minors of the Macaulay matrix vanish.** These minors are expressed by polynomial functions on X, hence $V_s \subset X$ is an algebraic variety.

Step 3: Clearly, $V = \bigcap V_i$. However, an intersection of Zariski closed subsets is Zariski closed, hence V is algebraic. ∎