# Geometria Algébrica I

## lecture 24: Hilbert polynomial and semicontinuity of dimension

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**IMPA, sala 232**

**November 16, 2018**

## Hilbert function and Policaré series

**DEFINITION:** Let $A_* = \bigoplus_{i\geqslant 0} A_i$ be a graded ring of finite type (that is, with each graded component finite-dimensional, and $A_0 = \mathbb{C}$), and $M_*$ a graded $A_*$-module, that is, a module $M_* = \bigoplus_{i\geqslant 0} M_i$ such that $A_d M_e \subset M_{d+e}$. **Hilbert function** is defined as $h(n) := \dim M_n$, for all $n \in \mathbb{Z}^{\geqslant 0}$. **Poincaré series** is generating series for the Hilbert function, $P_{M_*}(t) := \sum_{i\geqslant 0} h(i)t^i$.

## Policare series are rational

**THEOREM:** Let $A_*$ be a ring of finite type generated by $A_1$, and $M_*$ a finitely generated $A_*$-module. **Then its Poincaré series is a rational function of form $P(t) = \frac{Q(t)}{(1-t)^k}$, where $Q(t)$ is a polynomial.**

**Proof. Step 1:** Let $x \in A_1$, and $L_x(m) := xm$ be the multiplication by $x$. Consider the exact sequence

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{L_x} M_{n+1} \longrightarrow C_{n+1} \longrightarrow 0,$$

where $K_*$ is kernel of the map $L_x$ and $C_*$ its cokernel. Then $\dim(K_n) - \dim(M_n) + \dim(M_{n+1}) - \dim C_{n+1} = 0$, giving $P_{M_*}(t)(1-t) = P_{C_*}(t) - tP_{K_*}(t)$.

**Step 2:** We shall use induction on $\dim A_1$. When $\dim A_1 = 0$, $A_* = \mathbb{C}$, and any finitely generated module $M_*$ is finite-dimensional. Then $P_{M_*}(t)$ is a polynomial.

**Step 3:** Let $B_* := A_*/(x)$. Since $L_x$ acts trivially on $C_*$ and on $K_*$, we can consider $C_*$ and $K_*$ as $B_*$-modules. By induction assumption, the corresponding Poincare series satisfy $P_{C_*}(t) = \frac{Q_C(t)}{(1-t)^k}$ and $P_{K_*}(t) = \frac{Q_K(t)}{(1-t)^k}$. Then

$$P_{M_*}(t) = \frac{P_{C_*}(t) - tP_{K_*}(t)}{1-t}$$

is a rational function of the same nature. ■

$(1-t)^{-r} = \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} t^k$

**REMARK:** We assume $\binom{n}{-1} = 0$ for $n \geqslant 0$ and $\binom{-1}{-1} = 1$.

**LEMMA:** $(1-t)^{-r} = \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} t^k \qquad (*)$.

**Proof. Step 1:** Derivative of $(1-t)^{-r}$ is equal to $r(1-t)^{-r-1}$. On the other hand, derivative of the right hand side of (*) gives

$$\frac{d}{dt}\sum_{k=0}^{\infty}\binom{r+k-1}{r-1}t^k = \sum_{k=0}^{\infty} k\binom{r+k-1}{r-1}t^{k-1} \overset{l:=k-1}{=\!=\!=} \sum_{l=0}^{\infty}(l+1)\binom{r+l}{r-1}t^l =$$
$$= \sum_{l=0}^{\infty}(l+1)\frac{(r+l)!}{(r-1)!(l+1)!}t^l = \sum_{l=0}^{\infty}\frac{(r+l)!}{(r-1)!(l)!}t^l = r\sum_{l=0}^{\infty}\binom{r+l}{r}t^l$$

Denoting the right hand side of (*) by $V_r(t)$, we obtain that $V_r'(t) = rV_{r+1}(t)$.

**Step 2:** Assume that (*) is true: $(1-t)^{-r} = V_r(t)$. Applying derivative to both terms, we obtain $r(1-t)^{-r-1} = rV_{r+1}(t)$, which gives us (*) for $r+1$. Now (*) follows from induction in $r$. ■

## Hilbert polynomial

**THEOREM:** Let $A_*$ be a ring of finite type generated by $A_1$, and $M_*$ a finitely generated $A_*$-module. **Then there exists a polynomial $H(n)$ such that the Hilbert function $h(n)$ satisfies $h(n) = H(n)$ for all $n \gg 0$.**

**Proof:** Let $P(t) := \sum_{i\geqslant 0} h(i)t^i$ be the Poincaré series of $M_*$. Then $P(t) = \frac{Q(t)}{(1-t)^k}$, where $Q(t) = \sum_{i=0}^n a_n t^n$ is a polynomial, giving

$$P(t) = \sum_{i=0}^n a_i t^i \sum_{k=0}^\infty \binom{r+k-1}{r-1} t^k.$$

For each $k$, this gives

$$h(k) = \sum_{i=0}^n a_i \binom{r+k-i-1}{r-1} \qquad (**)$$

When $k > i$, the binomial coefficients

$$\binom{r+k-i-1}{r-1} = \frac{(k-i+1)(k-i+2)(k-i+3)...(k-i+r-1)}{(r-1)!}$$

are degree $r-1$ polynomials in $k$, hence the sum (**) is also polynomial. ■

**DEFINITION:** In these assumtions, $H(n)$ is called **the Hilbert polynomial of the module $M_*$.**

## Hilbert polynomial of the ring of polynomials

**CLAIM:** Let $A_*$ be a graded ring, $P(t)$ its Poincaré series, and $H(n)$ be its Hilbert polynomial. Write $P(t) = \frac{Q(t)}{(1-t)^d}$, where $Q(1) \neq 0$. **Then** $\deg H(n) = d-1$**.**

**Proof:** $H(n)$ is a sum of several terms proportional to $(n-i+1)(n-i+2)(n-i+3)...(n-i+d-1)$, which is a polynomial of of degree $d-1$. ■

**COROLLARY:** Let $A_* = \mathbb{C}[z_1, ..., z_d]$ be the ring of polynomials. Then its Poincaré series satisfies $P_d(t) = (1-t)^{-d}$. **In particular, its Hilbert polynomial has degree** $d$**.**

**Proof:** Consider the exact sequence

$$0 \longrightarrow K_n \longrightarrow A_n \xrightarrow{L_{z_d}} A_{n+1} \longrightarrow C_{n+1} \longrightarrow 0.$$

Clearly, $K_* = 0$ and $C_* = \mathbb{C}[z_1, ..., z_d]/(z_d) = \mathbb{C}[z_1, ..., z_{d-1}]$. Then $P_{A_*}(t)(1-t) = P_{C_*}(t)$ as shown above, which gives $P_d(t) = P_{d-1}(1-t)^{-1}$. However, $P_1(t) = 1+t+t^2+...$, giving $P_1(t) = (1-t)^{-1}$. Then $P_d(t) = (1-t)^{-d}$. ■

## Hilbert polynomial and dimension

**THEOREM:** Let $A_*$ be the graded ring of a projective variety $X$, and $d$ the degree of its Hilbert polynomial $H_{A_*}(n)$. **Then** $d = \dim X + 1$.

**Proof. Step 1:** Let $M_*$ be a finitely generated graded $A_*$-module with generators of degree $s_1, ..., s_r$. Then $h_{M_*}(n) \leqslant \sum_{i=1}^r h_{A_*}(n - s_i)$. **Therefore,** $\deg H_{M_*}(n) \leqslant \deg H_{A_*}(n)$**.**

**Step 2:** Let $A_* \subset B_*$, with $B_*$ finitely generated as an $A_*$-module. Then $h_{B_*}(n) \geqslant h_{A_*}(n)$, hence $\deg H_{A_*}(n) = \deg H_{B_*}(n)$. Therefore, for any finite dominant $\mathbb{C}^*$-invariant morphism $C(X) \longrightarrow C(Y)$ of the corresponding cone varieties, **the degrees of their Hilbert polynomials are equal.**

**Step 3:** Let $C_+(X) := C(X) \cup \{0\}$. We consider $C_+(X)$ as an affine subvariety of $\mathbb{C}^n$. By Noether normalization lemma, for an appropriate linear projection $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^d$, the morphism $\pi : C_+(X) \longrightarrow \mathbb{C}^d$ is finite and dominant. Then $\deg H_{A_*}(n) = \deg H_{\mathcal{O}_{\mathbb{C}^d}}(n) = d$. **However,** $C_+(X)$ **is also** $d$**-dimensional, because it admits a finite, dominant map to** $\mathbb{C}^d$**.**

**Step 4:** We have shown that $\deg H_{A_*}(n) = \dim C(X)$. However, $\dim C(X) = \dim X + 1$, because $C(X)$ **is fibered over** $X$ **with 1-dimensional fiber** $\mathbb{C}^*$**.** ■

## Semicontinuity

**DEFINITION:** For any sequence $\{x_i\}$ of points in a topological space $X$, **a limit point of** $\{x_i\}$ is any point in $x \in X$ such that any neighboorhood of $x$ contains infinitely many elements of the sequence. For any sequence $\{x_i\} \subset \mathbb{R}$ **limit superior** $\limsup x_i$ is supremum of all its limits, and **limit inferior** $\liminf x_i$ is inferior of all its limits.

**DEFINITION:** Let $X$ be a topological space, and $\varphi: X \longrightarrow \mathbb{R}$ a function. The function $\varphi$ is called **upper semi-continuous** if for any sequence $\{x_i\} \subset X$ and any limit point $x \in \lim x_i$, one has $\varphi(x) \geqslant \liminf(\varphi(x_i))$, and **lower semi-continuous** if $\varphi(x) \leqslant \limsup(\varphi(x_i))$.

**EXAMPLE: The raindrop function** $\xi: \mathbb{R} \longrightarrow \mathbb{Q}$ vanishes on irrational numbers, and satisfies $\xi(p/q) = 1/q$ for any irreducible fraction $p/q \in \mathbb{Q}$. Then **$\xi$ is upper semicontinuous**.

## Semicontinuity and preimages

**CLAIM:** Let $\varphi : X \longrightarrow \mathbb{R}$ be function. **Then $\varphi$ is upper semicontinuous if and only if $\varphi^{-1}([a, \infty[)$ is closed for all $a$.**

**Proof. Step 1:** Suppose that $\varphi$ is upper semicontinuous. Let $\{x_i\}$ be a sequence of points in $\varphi^{-1}([a, \infty[)$. Then $\varphi(x) \geqslant a$ for any $x \in \lim x_i$, hence $\varphi^{-1}([a, \infty[)$ is closed.

**Step 2:** Conversely, suppose that $\varphi^{-1}([a, \infty[)$ is closed for all $a$. Then for any $\{x_i\} \in \varphi^{-1}([a - \varepsilon, \infty[)$, any limit point $x$ satisfies $\varphi(x) \geqslant a - \varepsilon$. Therefore, for any sequence $x_i$ such that $\lim \varphi(x_i) \geqslant a$, any limit point $x$ satisfies $\varphi(x) \geqslant a$. ■

## Semicontinuity in Zariski topology

**DEFINITION:** Let $A$ be an algebraic variety. **Constructible set** is a subset $A_1 \subset A$ obtained from Zariski closed sets by taking complements, finite unions and finite intesections.

**PROPOSITION:** Let $X$ be an algebraic variety with Zariski topology, and $\varphi: \ X \longrightarrow \mathbb{Z}^{\geqslant 0}$ an upper semicontinuous function. **Then $\varphi$ takes finitely many values, and for each $t$, the set $\varphi^{-1}(t)$ is constructible.**

**Proof:** As shown above, the set $Z_a := \varphi^{-1}([a, \infty[)$ is Zariski closed. This gives a chain of embedded Zariski closed varieties parametrized by $a \in \mathbb{Z}^{\geqslant 0}$. Any decreasing chain of Zariski closed varieties stabilizes, hence for any sequence $a_1 \leqslant a_2 \leqslant ...$, only finitely many $Z_{a_i}$ are distinct. The sets

$$\varphi^{-1}(t) = \varphi^{-1}([t, \infty[) \backslash \varphi^{-1}([t+1, \infty[)$$

are constructible for all $t$. ■

The main result of today's lecture:

**THEOREM: (Semicontinuity of dimension)**
Let $Z \subset \mathbb{C}P^n \times X$ be an algebraic subvariety, and $\pi: \ \mathbb{C}P^n \times X \longrightarrow X$ the projection. For any $x \in X$, define $Z_x := Z \cap \pi^{-1}(x)$. **Then $\dim Z_x$ is an upper semicontinuous function of $x$.**

## Macaulay matrices

**DEFINITION:** Let $W \subset \mathbb{C}[z_0, ..., z_n]^d$ be a subspace in the space of homogeneous polynomials generated by $w_1, ..., w_r$, and $M^s_W : \mathbb{C}[z_0, ..., z_n]^{s-d} \otimes W \longrightarrow \mathbb{C}[z_0, ..., z_n]^s$ be the multiplication map. Then $M^s_W$ is called **the Macaulay matrix**. The rows of this matrix are enumerated by the pairs $(w_i, \theta)$, where $\theta$ is a monomial of degree $n-d$, and the columns by monomials $\xi$ of degree $n$. If we write $w_i\theta = \sum \alpha^{i,\theta}_j \xi_j$, the entry of $M^s_W$ corresponding to $((w_i, \theta), \xi)$ is given by $\alpha^{i,\theta}_j$.

**CLAIM:** Let $I^* \subset \mathbb{C}[z_0, ..., z_n]$ be a graded ideal generated by $W \subset \mathbb{C}[z_0, ..., z_n]^d$. **Then** $\dim I^s = \operatorname{rk} M^s_W$.

## Macaulay matrices and Hilbert function

**CLAIM:** Let $X$ be an affine variety, and $w_1, ..., w_r : X \longrightarrow \mathbb{C}[z_0, ..., z_n]^d$ functions with values in the space $\mathbb{C}[z_0, ..., z_n]^d$ of homogeneous polynomials. Let $I_*$ be the ideal generated by $w_1(x), ..., w_r(x)$, $A_* := \mathbb{C}[z_0, ..., z_n]/I_*$ the quotient ring, and $h_x(s)$ its Hilbert function. **Then $h_x(s)$ is upper semicontinuous as a function of $x \in X$,** for all $s \in \mathbb{Z}^{>0}$.

**Proof:** Since $\dim I^s = \operatorname{rk} M_W^s$, one has $h_x(s) = \dim \mathbb{C}[z_0, ..., z_n]^s - \operatorname{rk} M_W^s$. The number $\operatorname{rk} M_W^s$ is rank of maximal non-degenerate minor $\mathfrak{M}$ in the matrix $M_W^s$. This minor is non-zero in a Zariski open subset $U_{\mathfrak{M}}$, hence the set $\{x \in X \ | \ \operatorname{rk} M_W^s \geqslant w\}$ is Zariski open for all $w \in \mathbb{Z}^{>0}$. Then the set $\{x \in X \ | \ h_x(s) \geqslant w\}$ is closed. ■

## Semicontinuity of Hilbert polynomial

**COROLLARY:** Let $X$ be an affine variety, and $w_1, ..., w_r : X \longrightarrow \mathbb{C}[z_0, ..., z_n]^d$ functions with values in the space $\mathbb{C}[z_0, ..., z_n]^d$ of homogeneous polynomials. Let $I_*$ be the ideal generated by $w_1(x), ..., w_r(x)$, $A_* := \mathbb{C}[z_0, ..., z_n]/I_*$ the quotient ring, $h_x(s)$ its Hilbert function, and $H_x(t)$ its Hilbert polynomial. Then **the degree of the Hilbert polynomial $H_x(t)$ is also upper semicontinuous as a function of $x \in X$.**

**Proof. Step 1:** Let $\{x_i\}$ be a sequence of points such that $\deg H_{x_i}(t) \geqslant d$, and $x$ its limit. We need to show that $\deg H_x(t) \geqslant d$. Passing to a subsequence, we may assume that $\deg H_{x_i}(t) = d$. **Passing to a subsequence again, we may also assume that for each $t \in \mathbb{Z}^{>0}$, the sequence $h_{x_1}(t), h_{x_2}(t), ...$ stabilizes to $h(t) \in \mathbb{Z}$;** indeed, $h_x(t)$ as a function of $x$ takes only finitely many values, by semicontinuity. Then the sequences $h_{x_i}(1), h_{x_i}(2), ...h_{x_i}(t), ...$ and $h(1), h(2), ...h(t), ...$ are equal outside of finitely many $t$'s, and the corresponding Hilbert polynomials $H_{x_i}(t) = H(t)$ are equal.

**Step 2:** If the Hilbert polynomial $H_x(t)$ has smaller degree than $H(t)$, one would have $H_x(t) < H(t)$ for almost all $t$, but this is impossible because $H_x(t) = h_x(t)$ and $h(t) = H(t)$ for almost all $t$, and $h(t) \leqslant h_x(t)$ for all $t$ by semicontinuity of $h_x(t)$. ■

## Semicontinuity of dimension

**THEOREM: (Semicontinuity of dimension)**
Let $Z \subset \mathbb{C}P^n \times X$ be an algebraic subvariety, and $\pi : \mathbb{C}P^n \times X \longrightarrow X$ the projection. For any $x \in X$, define $Z_x := Z \cap \pi^{-1}(x)$. **Then $\dim Z_x$ is an upper semicontinuous function of $x$.**

**Proof:** Since the statement is local in $X$, we may assume that $X$ is affine. Then $Z$ is defined by a graded ideal $I^* \subset R := \mathcal{O}_X[z_0, ..., z_n]$. Let $w_1, ..., w_r$ be generators of the ideal $I_*$. Multiplying $w_i$ by all appropriate monomials if necessary, we may assume that $\deg w_i = d$. We can consider $w_1, ..., w_r$ as regular functions on $X$ with values in the space $\mathbb{C}[z_0, ..., z_n]^d$ of homogeneous polynomials of degree $d$. Then $Z_x$ is the set of common zeros of $w_1(x), ..., w_r(x) \in \mathbb{C}[z_0, ..., z_n]^d$. Therefore, the degree of the Hilbert polynomial of the ring $\frac{\mathbb{C}[z_0,...,z_n]}{\langle w_1(x),...,w_r(x)\rangle}$ is equal to $\dim Z_x$. We reduced semicontinuity of dimension to the following proposition, which is already proven.

**CLAIM:** Let $X$ be an affine variety, and $w_1, ..., w_r : X \longrightarrow \mathbb{C}[z_0, ..., z_n]^d$ functions with values in the space $\mathbb{C}[z_0, ..., z_n]^d$ of homogeneous polynomials. Let $I_*$ be the ideal generated by $w_1(x), ..., w_r(x)$, $A_* := \mathbb{C}[z_0, ..., z_n]/I_*$ the quotient ring, and $H_x(s)$ its Hilbert polynomial. **Then $\deg H_x(s)$ is upper semicontinuous as a function of $x$.** ■