Geometria Algébrica I

lecture 25: Constructible sets

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Constructible sets

DEFINITION: Let A be an algebraic variety. **Constructible set** is a subset $A_1 \subset A$ obtained from Zariski closed sets by taking complements, finite unions and finite intesections.

PROPOSITION: **Any constructible subset of X can be obtained as a finite union of sets of form $Z \backslash Z_1$,** where $Z \supset Z_1$ are Zariski closed.

Proof: A complement of such set can be obtained as $(X \backslash Z) \cap Z_1$. Intersection of two such sets is $(Z \backslash Z_1) \cap (Z' \backslash Z'_1) = Z \cap Z' \backslash (Z_1 \cup Z'_1)$. ■

CLAIM: A set $Z \subset X$ is constructible **if and only if for some affine cover $\{U_i\}$, the intersections $Z \cap U_i$ are constructible.**
Proof: Indeed, $Z = \bigcup Z \cap U_i$. ■

The main result of today's lecture

THEOREM: (Chevalley)
Let $\varphi : X \longrightarrow Y$ be a morphism of algebraic varieties iver $\mathbb{C}$, and $Z \subset X$ is a constructible set. **Then $\varphi(Z)$ is constructible.**

Reducing Chevalley theorem

THEOREM: (Chevalley)
Let $\varphi: X \longrightarrow Y$ be a morphism of algebraic varieties iver $\mathbb{C}$, and $Z \subset X$ is a constructible set. **Then $\varphi(Z)$ is constructible.**

Proof. Step 1: Let $\{U_i\}$ be an affine cover of X. Then $\varphi(Z) = \bigcup \varphi(Z \cap U_i)$. **Therefore, it suffices to prove Chevalley theorem assuming that X is affine.** Since the statement is local in Y, we can also assume that Y is affine. Replacing $X \subset \mathbb{C}^n$ by $\mathbb{C}^n$, we can always assume that $X = \mathbb{C}^n$. Indeed, if Z is constructible in X, it is also constructible in $\mathbb{C}^n$.

Step 2: Replacing Y by the union of its irreducible components, we may always assume that Y is irreducible.

Step 3: Let $\Gamma_Z \subset \mathbb{C}^n \times Y$ be the subset given by $\Gamma_Z := \{(x,y) \mid x \in Z, y = \varphi(x)\}$. Since Γ_Z is an intersection of the graph of φ and $Z \times Y$, it is constructible. Now, $\varphi(Z) = \pi(\Gamma_Z)$, where $\pi: \mathbb{C}^n \times Y \longrightarrow Y$ is the projection. **Therefore, we can always assume that $\varphi: \mathbb{C}^n \times Y \longrightarrow Y$ is the projection π, an $Z \subset \mathbb{C}^n \times Y$.**

Step 4: Representing π as a composition of the projections $\mathbb{C}^k \times Y \longrightarrow \mathbb{C}^{k-1} \times Y$, **we reduce the theorem to the case $Z \subset \mathbb{C} \times Y$ and $\varphi: \mathbb{C} \times Y \longrightarrow Y$ being the projection.**

Chevalley theorem reduced further

We reduced Chevalley theorem to the following situation.
THEOREM: Let Y be an irreducible affine variety, $\pi: \mathbb{C}\times Y \longrightarrow Y$ projection, and $Z \subset \mathbb{C}\times Y$ a constructible set. **Then $\pi(Z)$ is constructible.**

Proof. Step 1: We use induction in $\dim Y$. First, suppose that $\pi(Z)$ is not Zariski dense in Y. Then $\pi(Z)$ is contained in a proper Zariski closed subvariety $Y_0 \subset Y$. Replacing Y by Y_0 and using the induction assumption, we prove the statement of the theorem. Therefore, **we may assume that $\pi(Z)$ is Zariski dense in Y.**

Step 2: Suppose that $\pi(Z)$ contains a nonempty Zariski open subset $U \subset Y$, with $Y\backslash U = Y_0$. Then $\pi(Z) = U \cup \pi(\pi^{-1}(Y_0)\cap Z)$. The set $\pi^{-1}(Y_0)\cap Z$ is constructible in $\mathbb{C}\times Y_0$. Projecting it to Y_0 and using the induction assumption again, we obtain that $\pi(\pi^{-1}(Y_0)\cap Z)$ is constructible. Therefore, the same is true for $\pi(Z)$. It remains to prove that **$\pi(Z)$ contains a nonempty Zariski open subset.** We reduced Chevalley theorem to the following proposition.

PROPOSITION: Let Y be an irreducible affine variety, $\pi: \mathbb{C}\times Y \longrightarrow Y$ projection, and $Z \subset \mathbb{C}\times Y$ a constructible set. Suppose that $\pi(Z)$ is Zariski dense in Y. **Then $\pi(Z)$ contains a nonempty Zariski open subset of Y.**

Proof: Later today.

Symmetric polynomials (reminder)

DEFINITION: Symmetric polynomial $P(z_1, ..., z_n) \in \mathbb{C}[z_1, ..., z_n]$ is a polynomial which is invariant with respect to the symmetric group Σ_n acting on $\mathbb{C}[z_1, ..., z_n]$ by permutations.

DEFINITION: Consider the polynomial $P(z_1, ..., z_n, t) := \prod_{i=1}^n (t + z_i) = \sum e_i t^i$, with $e_i \in \mathbb{C}[z_1, ..., z_n]$. Then e_i are called **elementary symmetric polynomials** on $z_1, ..., z_n$.

THEOREM: Every symmetric polynomial on $z_1, ..., z_n$ can be polynomially expressed through the elementary symmetric polynomials.

Proof: Left as an exercise. ■

Resultants

DEFINITION: Let $P(t), Q(t)$ be polynomials, α_i roots of $P(t)$, β_j roots of $Q(t)$, and $R(P,Q) := \prod_{i,j}(\alpha_i - \beta_j)$. Since $R(P,Q)$ is invariant under permutations of α_i and of β_j, it can be expressed polynomially through the elementary symmetric polynomials on α_i and β_j, that is, on coefficients of $P(t), Q(t)$. The quantity $R(P,Q)$, expressed as polynomial on the coefficients of $P(t)$ and $Q(t)$ is called **the resultant** of $P(t), Q(t)$.

CLAIM: Let $P, Q \in k[t]$ be polynomials over a field. **Then $R(P,Q) = 0$ if and only if $P(t)$ and $Q(t)$ have a common root over the algebraic closure $\overline{k}$.**

■

Proposition 1: Let Y be an irreducible affine variety, and $P(t), Q(t) \in \mathcal{O}_Y[t]$ polynomials defining subvarieties $A, B \subset \mathbb{C} \times Y$. Assume that $\pi(A \cap B)$ is Zariski dense in Y. **Then $R(P,Q) = 0$,** and $P(t), Q(t)$ have a common multiplier over $k(Y)[t]$.

Proof: Let $Y_0 \subset Y$ be the subvariety defined by $R(P,Q) = 0$. Then $\pi(A \cap B) \subset Y_0$. **Since $\pi(A \cap B)$ is Zariski dense, $R(P,Q) = 0$.** ■

Resultants and subvarieties in $Y \times \mathbb{C}$

COROLLARY: Let Y be an irreducible affine variety, and $P_1(t), P_2(t), ...P_r(t) \in \mathcal{O}_Y[t]$ be polynomials defining subvarieties $A_1, A_2, ...A_r \subset \mathbb{C} \times Y$. Suppose that $\pi(A_1 \cap A_2 \cap ... A_r)$ is Zariski dense in Y. **Then $P_i(t)$ have a common multiplier over $k(Y)[t]$.**

Proof: By Proposition 1, $P_1(t)$ and $P_2(t)$ have a common multiplier $\tilde{Q}(t)$ over $k(Y)[t]$. Multiplying $\tilde{Q}(t)$ by the product of appropriate denominators $s \in \mathcal{O}_Y$, we obtain $Q(t) = s\tilde{Q}(t) \in \mathcal{O}_Y[t]$. Then $sP_1(t)$ and $sP_2(t)$ divides $Q(t)$, hence $A_1 \cap A_2 \subset V(s) \cup V(Q)$, where $V(\cdot)$ denotes the zero set. Therefore, the projection of the common zero set of $Q, P_3(t), ..., P_r(t)$ is Zariski dense in Y. Using induction on r, we obtain that all $P_i(t)$ have a common multiplier.
■

Corollary 1: Let Y be an irreducible affine variety, and $R \subset \mathbb{C} \times Y$ a subvariety such that the projection $\pi(R)$ is Zariski dense in Y. **Then there exists a polynomial $P(t) \in \mathcal{O}_Y[t]$ vanishing in R such the zero set of $P(t)$ is equal to R outside of a proper Zariski closed subvariety $\pi^{-1}(Y_0)$.** ■

Constructible subsets in $Y \times \mathbb{C}$

Proposition 2: Let Y be an irreducible affine variety, and $R_1 \subset R \subset \mathbb{C} \times Y$ subvarieties. Suppose that R is the sero set of a polynomial $P(t) \in \mathcal{O}_Y[t]$. Then there exists **an open subset $U \subset Y$ and polynomial $Q(t) \in \mathcal{O}_U[t]$. dividing $P(t)$ in $\mathcal{O}_U[t]$ such that the zero set of $Q(t)$ is equal to $R_1 \cap \mathbb{C} \times U$.**

Proof. Step 1: By Corollary 1, there exists $U \subset Y$ and polynomial $Q(t) \in \mathcal{O}_U[t]$ such that its zero set is $R_1 \cap \mathbb{C} \times U$. Replacing $Q(t)$ by a polynomial of smaller degree, we can assume that $Q(t)$ is reduced (has no multiple roots over the algebraic closure). Then $\mathcal{O}_U[t]/(Q)$ is embedded to $k(U)[t]/(Q)$, which is a direct sum of fields, hence the ideal (Q) is radical.

Step 2: Since $R_1 \subset R$, one has $\operatorname{Ann}(R) \subset \operatorname{Ann}(R_1) = (Q)$. Therefore, $P(t)$ divides $Q(t)$. ■

We shall also need the following lemma.

Lemma 1: Let Y be an irreducible affine variety, and $Z \subset \mathbb{C} \times Y$ the zero set of a polynomial $P(t) \in \mathcal{O}_Y[t]$. **Then the projection $\pi(Z)$ contains a nonempty Zariski open subset of Y.**

Proof: Write $P(t) = a_0 + a_1 t + ... + a_n t^n$, with $a_i \in \mathcal{O}_Y$, and let $U \subset Y$ be the complement to the zero set of a_n. Then the projection $Z \cap \pi^{-1}(U) \xrightarrow{\pi} U$ is finite and dominant, hence surjective. ■

Chevalley theorem: finishing the proof

We reduced Chevalley theorem to the following

PROPOSITION: Let Y be an irreducible affine variety, $\pi : \mathbb{C} \times Y \longrightarrow Y$ projection, and $Z \subset \mathbb{C} \times Y$ a constructible set. Suppose that $\pi(Z)$ is Zariski dense in Y. **Then $\pi(Z)$ contains a nonempty Zariski open subset of Y.**

Proof. Step 1: As shown above, Z is a union of several subsets of form $Z_1 \backslash Z_2$, where Z_i are Zariski closed. Then for at least one of these subsets the image $\pi(Z_1 \backslash Z_2)$ is Zariski dense. **Replacing Z by this subset, we may assume $Z = Z_1 \backslash Z_2$, where $Z_2 \subset Z_1$.**

Step 2: Replacing Y by an open subset again, we may assume that Z_1 and Z_2 are zero set of polynomials $P_1(t)$ and $P_2(t) \in \mathcal{O}_U[t]$ (Corollary 1). Replacing U by a smaller open subset again, we may assume that $P_1(t)$ divides $P_2(t)$ (Proposition 2). Let $Y_1 \subset U$ be the subset defined by the resultant $R(P_1, P_2)$. Replacing U by $U \backslash Y_1$, we may assume that $P_1(t), P_2(t)$ have no common zeros in $\mathbb{C} \times U$. **Then Z is a Zariski closed subset, defined by an ideal $Q(t) = \frac{P_1(t)}{P_2(t)}$.**

Step 3: Now, Lemma 1 implies that $\pi(Z)$ contains a nonempty Zariski open subset. ■