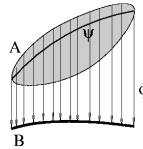


Home assignment 1: Zorn lemma

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solutions with your monitor. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

Definition 1.1. Let $\phi : A \rightarrow B$ be a surjective map of sets. A **section** of ϕ is a map $\psi : B \rightarrow A$ such that $\psi \circ \phi = \text{Id}$.



Definition 1.2. Axiom of Choice (AC) states that each surjective map admits a section.

Definition 1.3. Partial order is a relation $x \prec y$, which is **transitive** (if $x \prec y$ and $y \prec z$ then $x \prec z$) and **non-reflexive** ($z \prec z$ does not hold for any z). A set with partial order is called **partially ordered set**, or **poset**.

Definition 1.4. A partial order on S is called **total order** if for all $x \neq y$ either $x \prec y$ or $y \prec x$. A totally ordered poset S is called **well ordered** if any subset $S_1 \subset S$ has a minimal element (an element $v \in S_1$ such that $v \prec v'$ for all $v' \in S_1$ distinct from v).

Definition 1.5. Two posets A, B are called **isomorphic** if there exists a bijection from A to B preserving the order, with the inverse also preserving the order. Isomorphism classes of well ordered sets are called **ordinals**, or **ordinal numbers**. **Finite ordinals** are the same as natural numbers.

Remark 1.1. One can **add** the ordinals by taking a union $X \amalg Y$ of two well ordered sets and setting $X \prec Y$. Also, the ordinals can be multiplied: order on a product

$$(x, y) \prec (x', y') \text{ if } x \prec x', \text{ or } x = x', y \prec y'.$$

Exercise 1.1 (*). Prove that addition of ordinals is non-commutative.

Exercise 1.2. Prove that addition of ordinals is associative.

Exercise 1.3 (*). Prove that multiplication of ordinals is non-commutative.

Definition 1.6. Interval in a totally ordered set S is $[a, b] := \{x \in S \mid a \prec x \prec b\}$ **Initial interval** is an interval starting from the minimal element.

Exercise 1.4 (!). Let X, Y be well ordered sets. Prove that X is isomorphic to an initial interval of Y , or Y is isomorphic to an initial interval of X .

Exercise 1.5. Prove that such an isomorphism is unique.

Definition 1.7. Zermelo theorem states that any set admits well ordering.

Exercise 1.6. Deduce the Axiom of Choice from the Zermelo theorem.

Hint. The section ψ will map $b \in B$ to the minimal element in $\phi^{-1}(b)$.

Definition 1.8. Let (S, \prec) be a poset. An element $x \in S$ is called **maximal** if there is no $y \in S$ such that $x \prec y$. For a subset $S_1 \subset S$ and $x \in S$, we write $S_1 \prec x$ if $\xi \prec x$ for all $\xi \in S_1$. **Zorn lemma** states that any poset (S, \prec) has a maximal element if for any well ordered subset $S_1 \subset S$ there exists $x \in S_1$ such that $S_1 \setminus \{x\} \prec x$.

Exercise 1.7 (!). Deduce Zermelo theorem from Zorn lemma.

Hint. Let A be a set which we want to endow with a well ordering. Take for S the set of subsets of A equipped with well ordering, and write $S_1 \prec S_2$ if S_1 is an initial interval of S_2 . Prove that the poset S satisfies assumptions of Zorn lemma, and any maximal element of S is A with well ordering.

Exercise 1.8 (!). Let A be a set, and S the set of all subsets $A_0 \subset A$ equipped with a well ordering \prec_{A_0} . Assume that S does not contain A . We intend to deduce Zermelo theorem from Axiom of Choice by absurd.

- Using Axiom of Choice, construct a map $\phi : S \rightarrow A$ mapping $W \in S$ to an element x in the complement $A \setminus S$.
- Let $R \subset S$ be the set of all well ordered subsets $(W, \prec_W) \subset A$ such that for any initial interval $W_0 \subset W$, the minimal element of the complement $W \setminus W_0$ is $\phi(W_0)$. Prove that R is well ordered by inclusion.
- Prove that the union of all $W \in R$ is well ordered.

Exercise 1.9 (!). Deduce Zorn lemma from Axiom of Choice, as follows. Like for Zermelo theorem, we prove Zorn lemma ad absurdum. Let (A, \prec) be a poset without a maximal element, and S the set of well ordered subsets of A .

- Prove that for any well ordered $W \subset A$, there exists $x \in A$ which satisfies $W \prec x$. Using Axiom of Choice, find a function $\phi : S \rightarrow A$ such that $W \prec \phi(W)$ for all W .
- Let $R_\phi \subset S$ be the set of all well ordered subsets $W \subset A$ such that for any initial interval $W_0 \subset W$, the minimal element of $W \setminus W_0$ is $\phi(W_0)$. Prove that R is well ordered by inclusion.
- Prove that the union of all $W \in R$ is well ordered.