Home assignment 2: Applications of Axiom of Choice

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solution with your monitor. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

Exercise 2.1. Let A, B be two sets. Prove that A is equinumerous to a subset of B, or B is equinumerous to a subset of A.

Hint. Use Zermelo theorem.

Exercise 2.2 (!). (Cantor-Bernstein-Schroeder theorem)
Suppose that A is equinumerous to a subset of B, and B is equinumerous to a subset of A. Prove that A is equinumerous to B.

Remark 2.1. Cantor-Bernstein-Schroeder theorem is in fact independent from the Axiom of Choice. Please google its proof, if you never saw it, it is not very complicated, but hard to invent (Cantor spent 20 years and became very depressed trying to prove it).

Exercise 2.3 (!). Let X be an infinite set. Prove that $X \times \mathbb{Z}$ is equinumerous to X.

Hint. Use Zermelo theorem.

Exercise 2.4. Let $I \subset R$ be an ideal in a ring. Prove that I is contained in a maximal ideal.

Definition 2.1. Let V be a vector space. **Hamel basis**, or just **basis** in V is a maximal set of linearly independent vectors in V.

Exercise 2.5. Prove that any vector space has a basis.

Exercise 2.6. (!) Let S_1, S_2 be bases in a vector space W over a field k, and $A \in k^{S_1 \times S_2}$ the transition matrix expressing S_1 through S_2. Denote by $T \subset S_1 \times S_2$ the set of pairs of indices $\alpha \in S_1, \beta \in S_2$ such that the corresponding coefficient $a_{\alpha,\beta}$ in A is non-zero.

a. Prove that the projection functions $\pi_i : T \longrightarrow S_i$ are surjective and preimages $\pi_i^{-1}(x)$ are finite for any $x \in S_i$.

b. Prove that T is equinumerous to S_1 and to S_2.

Hint. When S_i are finite, use standard arguments. When they are infinite, construct injective maps $S_i \hookrightarrow T \hookrightarrow S_i \times \mathbb{Z}$, prove that $S_i \times \mathbb{Z}$ is equinumerous to S_i, and apply Cantor-Bernstein-Schroeder.