Home assignment 3: Noetherian rings

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

Definition 3.1. A ring is called **Noetherian** if any increasing chain of ideals $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq ...$ terminates. It is called **Artinian** if any decreasing chain of ideals $I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq ...$ terminates.

Exercise 3.1. Prove that the ring $\mathbb{Z}$ of integers is Noetherian.

Exercise 3.2. Prove that a localization of a Noetherian ring is Noetherian.

Exercise 3.3. Let R be a ring which has only one prime ideal. Is it necessarily Artinian?

Exercise 3.4. Construct a ring which is Artinian, but not Noetherian.

Exercise 3.5. Let $M = S^1$ be a circle, and $C(M)$ the ring of continuous functions on S^1.

a. (*) Prove that $C(M)$ is not Noetherian.

b. Prove that $C(M)$ is not Artinian.

Exercise 3.6. Prove that the ring $\mathbb{C}[[t]]$ of formal power series of one variable is Noetherian.

Exercise 3.7 (*). Prove that the ring $\mathbb{C}[[t_1, ..., t_n]]$ of formal power series is Noetherian.

Definition 3.2. Let R be a ring, and $\mathbb{Z}[\mathrm{Mod}]$ a free abelian group, formally generated by isomorphism classes of finitely generated R-modules. The **Grothendieck group** $K_0(R)$ is the quotient of $\mathbb{Z}[\mathrm{Mod}]$ by all relations of form $[M_2] - [M_1] - [M_0]$, where M_i fit into an exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$.

Exercise 3.8. Recall that an R-module is called **cyclic** if it is isomorphic to R/I, for some ideal I. Prove that $K_0(R)$ is generated by the classes of cyclic R-modules.

Exercise 3.9. Prove that $K_0(\mathbb{Z}) = \mathbb{Z}$.

Exercise 3.10 (*). Prove that $K_0(R) = \mathbb{Z}$ for any principal ideal ring R without zero divisors.

Exercise 3.11 (*). Let $u: \ M \longrightarrow M$ be a surjective endomorphism of a Noetherian R-module. Prove that it is injective.

Hint. Apply the terminating chain condition to the chain $\ker u \subset \ker u^2 \subset ....$

Exercise 3.12 (*). Let $R \subset \mathbb{C}[x, y]$ be a ring of polynomials $P(x, y)$ such that all derivatives $\frac{\partial^i}{\partial y^i}$ vanish in 0. Determine whether R is Noetherian or not.

Exercise 3.13. Assume that $R[t]$ is Noetherian. Prove that R is Noetherian, or find a counterexample.