Home assignment 6: Limits and colimits

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and essential for further work.

6.1 Colimits and germs

Definition 6.1. Let $\mathcal{C}$ be a commutative diagram of vector spaces A, B – vector spaces corresponding to two vertices of a diagram, and $a \in A, b \in B$ elements of these vector spaces. Write $a \sim b$ if a and b are mapped to the same element $d \in D$ by a composition of arrows from $\mathcal{C}$. Let $\sim$ be an equivalence relation generated by such $a \sim b$.

Exercise 6.1. a. Let $A \xrightarrow{\phi} B$ be a diagram of two spaces and one arrow. Prove that $b \sim b'$ is equivalent to $b = b'$ for each $b, b' \in B$.

b. Let $A \xrightarrow{\phi} B$, $A \longrightarrow 0$ be a diagram of three spaces, with ϕ injective. Prove that for each $b, b' \in B$, $b \sim b'$ is equivalent to $b - b' \in \operatorname{im}\phi$.

Definition 6.2. Let $\{C_i\}$ be a set of vector spaces associated with the vertices of a commutative diagram $\mathcal{C}$, and $E \subset \bigoplus_i C_i$ a subspace generated by the vectors $(x - y)$, where $x \sim y$. A quotient $\bigoplus_i C_i / E$ is called **a direct limit** of a diagram $\{C_i\}$. The same notion is also called **colimit** and **inductive limit**. Direct limit is denoted $\varinjlim$.

Exercise 6.2. Let $... \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ be a diagram with all arrows injective. Prove that $\varinjlim C_i$ is a union of all C_i.

Exercise 6.3. Let $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ... \longrightarrow C_n$ be a diagram. Prove that $\varinjlim C_i = C_n$.

Exercise 6.4. Find an example of a diagram $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ where all spaces C_i are non-zero, and the colimit $\varinjlim C_i$ vanishes.

Exercise 6.5 (*). Find an example of a diagram $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ where all spaces C_i are non-zero, all arrows are also non-zero, and the colimit $\varinjlim C_i$ vanishes.

Definition 6.3. A diagram $\mathcal{C}$ is called **filtered** if for any two vertices C_i, C_j, there exists a third vertex C_k, and sequences of arrows leading from C_i to C_k and from C_j to C_k.

Exercise 6.6. Let $\mathcal{C}$ be a commutative diagram of vector spaces C_i, with all C_i equipped with a ring structure, and all arrows ring homomorphisms. Suppose that the diagram $\mathcal{C}$ is filtered. Prove that $\varinjlim C_i$ admits a structure of a ring, such that the natural maps $C_i \longrightarrow \varinjlim C_i$ are ring homomorphisms.

Definition 6.4. Let M be a smooth manifold, or real analytic manifold, $x \in M$ its point, and $\{U_i\}$ the set of all its neighbourhoods. Consider a commutative diagram with vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. For each vertex U_i we take a vector space $C^\infty(U_i)$, $C^\omega(U_i)$ $C^0(U_i)$ of smooth (real analytic, continuous) functions on U_i, and for each arrow the corresponding restriction map. The direct limit of this diagram is called **the ring of germs of smooth (real analytic, continuous) functions in** x.

Remark 6.1. This limit is indeed a ring, as follows from the previous exercise.

Definition 6.5. A ring is called **local** if it contains only one maximal ideal.

Exercise 6.7 (!). Prove that the ring of germs of real analytic functions in $x \in \mathbb{R}^n$ is local. Prove that it has no zero divisors.

Exercise 6.8 (!). Prove that the ring of germs of smooth functions in $x \in \mathbb{R}^n$ is local. Prove that it has zero divisors.

Exercise 6.9 (*). Let A be the ring of germs of continuous functions in $x \in \mathbb{R}^2$. Prove that A has only one prime ideal, or find a counterexample.

Exercise 6.10 (**). Let A be the ring of germs of continuous functions in $x \in \mathbb{R}$. Prove that A has only one prime ideal, or find a counterexample.

6.2 Limits and completions

Definition 6.6. Let $\{C_i\}$ be a set of vector spaces or abelian groups associated with the vertices of a commutative diagram $\mathcal{C}$, and $V \subset \prod_i C^i$ the set of all collections $v_i \in C_i$ such that for any map $\phi_{ij} : C^i \longrightarrow C^j$ associated with a vertex, one has $\phi_{ij}(v_i) = v_j$. The space V is called **an inverse limit** of a diagram $\{C_i\}$. The same notion is also called **limit** and **projective limit**. Inverse limit is denoted $\lim\limits_{\leftarrow}$.

Exercise 6.11. Let $... \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ be a diagram with all arrows injective. Prove that $\lim\limits_{\leftarrow} C_i$ is an itersection of all C_i.

Exercise 6.12. Let $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ... \longrightarrow C_n$ be a diagram. Prove that $\lim\limits_{\leftarrow} C_i = C_1$.

Exercise 6.13 (*). Find an example of a diagram $... \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ where all spaces C_i are non-zero, and the limit $\lim\limits_{\leftarrow} C_i$ vanishes.

Definition 6.7. A diagram $\mathcal{C}$ is called **cofiltered** if for any two vertices C_i, C_j, there exists a third vertex C_k, and sequences of arrows leading from C_k to C_i and to C_j.

Exercise 6.14. Let $\mathcal{C}$ be a commutative diagram of vector spaces C_i, with all C_i equipped with a ring structure, and all arrows ring homomorphisms. Suppose that the diagram $\mathcal{C}$ is cofiltered. Prove that $\lim\limits_{\leftarrow} C_i$ admits a structure of a ring, such that the natural maps $\lim\limits_{\leftarrow} C_i \longrightarrow C_i$ are ring homomorphisms.

Definition 6.8. Let G be a topological group. A sequence $\{g_i\} \subset G$ is called **a Cauchy sequence** if for any neighbourhood of $U \ni e$ of unity there exists $g \in G$ such that gU contains almost all elements of $\{g_i\}$. Cauchy sequences $\{a_i\}, \{b_i\} \subset G$ are **equivalent** if the sequence $a_1, b_1, a_2, b_2, ...$ is Cauchy. **Completion** of a topological group is the set of all equivalence classes of Cauchy sequences.

Exercise 6.15. Define the following topology on the completion $\bar{G}$ of a topological group G. The base of open sets in $\bar{G}$ is obtained from the open sets $U \subset G$, with an open set $\bar{U} \subset \bar{G}$ defined as the set of all sequences $\{a_i\}$ such that almost all a_i belong to U. Define the multiplicative structure on $\bar{G}$ by $\{a_i\}\{b_i\} = \{a_i b_i\}$. Prove that a completion $\bar{G}$ of a topological group G is a topological group.

Exercise 6.16. Consider a map from a topological group G to its completion $\bar{G}$, taking $g \in G$ to the sequence $g, g, g, g, ...$ Prove that this defines a homomorphism $G \longrightarrow \bar{G}$ of topological groups, and G is dense in $\bar{G}$.

Definition 6.9. Let $I \subset R$ be an ideal in a ring. Define **the I-adic topology** on R with the base of open sets given as $a + I^k \subset R$, for all $a \in R$ and $k \in \mathbb{Z}^{>0}$. Define **I-adic completion** of R as the completion of $\frac{R}{\cap_k I^k}$ in this topology.

Exercise 6.17 (!). Prove that I-adic completion $\bar{R}$ of R is equipped with a ring structure in such a way that the natural map $R \longrightarrow \bar{R}$ is a homomorphism. Prove that $\bar{R}$ is isomorphic to the inverse limit $\lim\limits_{\leftarrow} R/I^k$.

Exercise 6.18. Define **the p-adic norm on** $\mathbb{Z}$ as the map $\nu_p : \mathbb{Z} \longrightarrow \mathbb{Q}$ taking $x \in \mathbb{Z}$ to p^{-n}, where n is the maximal number such that x is divisible by p^n in $\mathbb{Z}$. Consider the topology associated with the ideal (p) as in Definition 6.9, called **the p-adic topology**. Prove that the completion of $\mathbb{Z}$ with respect to the norm ν_p is equal to the completion of $\mathbb{Z}$ with respect to the p-adic topology.

Exercise 6.19. Let p be a prime number. Define **the ring $\mathbb{Z}_p$ of p-adic numbers** as $\lim\limits_{\leftarrow} \mathbb{Z}/p^k\mathbb{Z}$.

a. Prove that any power series $\sum_i = 0^\infty a_i p^i$ converges in $\mathbb{Z}_p$, for any $a_i \in \mathbb{Z}$. Prove that any element in $\mathbb{Z}_p$ can be obtained as a sum of such series.

b. Prove that $\mathbb{Z}_p$ is a local ring, and every $x \in \mathbb{Z}_p$ which is not divisible by p is invertible in $\mathbb{Z}_p$.

Exercise 6.20 (*). Let G be a group, and $\mathfrak{S}$ the set of all normal subgroups of finite index. **Profinite topology** on G is topology with the base of open sets given by all gS, where $g \in G$ and $S \in \mathfrak{S}$. **Profinite completion** is the completion of the quotient $\frac{G}{\bigcap_{S\in\mathfrak{S}} S}$ with respect to this topology. Prove that the profinite completion of G is $\lim\limits_{\leftarrow} G/S$, where the limit is taken over all $S \in \mathfrak{S}$.

Exercise 6.21 (*). Prove that the profinite completion of a group is compact and Hausdorff.

Exercise 6.22 (*). Prove that the profinite completion of $\mathbb{Z}$ is $\prod_p \mathbb{Z}_p$, where the product is taken over all prime numbers.

6.3 Limits and colimits in categories

Definition 6.10. An initial object of a category is an object $I \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(I,X)$ is always a set of one element. **A terminal object** is $T \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(X,T)$ is always a set of one element.

Definition 6.11. Let $S = \{X_i, \phi_{ij}\}$ be a commutative diagram in $\mathcal{C}$, and $\vec{\mathcal{C}}_S$ be a category of pairs (object X in $\mathcal{C}$, morphisms $\psi_i : X \longrightarrow X_i$, defined for all X_i) making the diagram formed by $(X, X_i, \psi_i, \phi_{ij})$ commutative.

$$
\begin{tikzcd}
X \arrow[dr, "\psi_1"] \arrow[drr, "\psi_3"] \arrow[ddr, "\psi_2"'] & & \\
& X_1 \arrow[r, "\phi_{12}"] \arrow[d, "\phi_{13}"] & X_2 \arrow[dl, "\phi_{23}"] \\
& X_3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "\Psi"] \arrow[drr, "\psi'_1"] \arrow[drrr, "\psi'_2"] \arrow[ddrr, "\psi'_3"'] & X \arrow[dr, "\psi_1"] \arrow[drr, "\psi_2"] \arrow[ddr, "\psi_3"] & & \\
& & X_1 \arrow[r, "\phi_{12}"] \arrow[d, "\phi_{13}"] & X_2 \arrow[dl, "\phi_{23}"] \\
& & X_3 &
\end{tikzcd}
$$

Morphisms $\mathcal{M}or(\{X,\psi_i\},\{X',\psi'_i\})$, are morphisms $\Psi \in \mathcal{M}or(X,X')$, making the diagram formed by $(X, X', \psi_i, \psi'_i, \phi_{ij})$ commutative. The terminal object in this category is called **limit**, or **inverse limit** of the diagram S. **Colimit**, or **direct limit** is obtained from the previous definition by inverting all arrows and replacing “terminal” by “initial”.

Exercise 6.23 (!). Let $S = \{X_i, \phi_{ij}\}$ be a commutative diagram of vector spaces. Prove that the colimit of S in the sense of Definition 6.2 coincides with the one defined in Definition 6.11.

Exercise 6.24 (!). Let $S = \{X_i, \phi_{ij}\}$ be a commutative diagram of vector spaces. Prove that the limit of S in the sense of Definition 6.6 coincides with the one defined in Definition 6.11.