Home assignment 7: Yoneda lemma

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and essential for further work.

Definition 7.1. Consider the functor $h_A : \mathcal{C} \longrightarrow \mathcal{S}ets$ taking $X \in \mathcal{O}b(\mathcal{C})$ to $\mathcal{M}or(A, X)$. We say that h_A **is represented by an object** $A \in \mathcal{O}b(\mathcal{C})$**.**

Definition 7.2. Let $F, G : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be functors. **A natural transformation of functors** from F to G is a morphism $\Psi_X : F(X) \longrightarrow G(X)$ such that for any $\phi \in \mathcal{M}or(X, Y)$, one has $F(\phi) \circ \Psi_Y = \Psi_X \circ G(\phi)$.

Exercise 7.1. Let $\Phi : h_A \longrightarrow F$ be a natural transformation of functors from $\mathcal{C}$ to sets, and $\lambda \in \mathcal{M}or(A, B)$. Consider the diagram

$$\begin{array}{ccc} h_A(A) = \mathcal{M}or(A, A) & \xrightarrow{f \mapsto f\circ\lambda} & h_A(B) = \mathcal{M}or(A, B) \\ \Phi_A \big\downarrow & & \big\downarrow \Phi_B \\ F(A) & \xrightarrow{F(\lambda)} & F(B) \end{array}$$

Prove that Φ_B takes $\lambda \in h_A(B)$ to $F(\lambda)(\Psi_A(\mathsf{Id}_A))$, where Id_A is considered as an element of $h_A(A)$.

Exercise 7.2. Let F, h_A be functors from $\mathcal{C}$ to sets,

a. Prove that for any $v \in F(A)$, there exists a natural transformation of functors $\Phi : h_A \longrightarrow F$ taking $\mathsf{Id}_A \in h_A(A)$ to $F(A)$.

b. Prove that Φ is unique.

Hint. Use the previous exercise.

Definition 7.3. Let $\mathcal{C}$ be a category and $\mathcal{C}^\circ$ be category with the same objects, $\mathcal{M}or_{\mathcal{C}^\circ}(A, B) = \mathcal{M}or_{\mathcal{C}}(B, A)$ and the inverted order of taking compositions. Then $\mathcal{C}^\circ$ is called **the opposite category** of $\mathcal{C}$.

Exercise 7.3. Consider a category $\mathcal{F}unc(\mathcal{C}, \mathcal{S}ets)$ from $\mathcal{C}$ to $\mathcal{S}ets$, with objects all functors from $\mathcal{C}$ to sets, and morphisms natural transforms.

a. Prove that $\mathcal{M}or_{\mathcal{F}unc(\mathcal{C},\mathcal{S}ets)}(h_A, h_B)$ is naturally identified with $\mathcal{M}or_{\mathcal{C}}(B, A)$.

b. Prove that the map associating h_A to each $A \in \mathcal{O}b(\mathcal{C})$ defines a functor from $\mathcal{C}$ to $\mathcal{F}unc(\mathcal{C}, \mathcal{S}ets)$.

c. Prove that this map defines an equivalence of $\mathcal{C}^\circ$ and the category of all representable functors $h_A \in \mathcal{O}b(\mathcal{F}unc(\mathcal{C}, \mathcal{S}ets))$.

Hint. Use the previous exercise.

Exercise 7.4. Prove that any category $\mathcal{C}$ is equivalent to the category $\mathcal{G}$ of contravariant functors $\mathcal{C}^\circ \longrightarrow \mathcal{S}ets$ representable by $h^\circ_A(X) := \mathcal{M}or(X, A)$.

Definition 7.4. **An initial object** of a category is an object $I \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(I, X)$ is always a set of one element. **A terminal object** is $T \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(X, T)$ is always a set of one element.

Exercise 7.5. Let $\mathcal{C}$ be a category, and I the set of one element.

a. Prove that the terminal object represents the functor $\mathcal{C}^\circ \longrightarrow \mathcal{S}ets$ taking any object of $\mathcal{C}$ to I and any morphism to Id_I.

b. Prove that the initial object represents the functor $\mathcal{C} \longrightarrow \mathcal{S}ets$ taking any object of $\mathcal{C}$ to I and any morphism to Id_I.

c. Prove that an initial and a terminal objects of a category are unique.

Definition 7.5. Let $X, Y \in \mathcal{O}b(\mathcal{C})$. Consider the functor from $\mathcal{C}^\circ$ to $\mathcal{S}ets$ mapping $Z \in \mathcal{O}b(\mathcal{C})$ to $\mathcal{M}or(Z, X) \times \mathcal{M}or(Z, Y)$. An object of $\mathcal{C}$ representing this functor is called **the product of** X **and** Y, denoted $X \times Y$.

Definition 7.6. Let $X, Y \in \mathcal{O}b(\mathcal{C})$. Consider the functor from $\mathcal{C}$ to $\mathcal{S}ets$ mapping $Z \in \mathcal{O}b(\mathcal{C})$ to $\mathcal{M}or(X, Z) \times \mathcal{M}or(Y, Z)$. An object of $\mathcal{C}$ representing this functor is called **the co product of** X **and** Y, denoted $X \coprod Y$.

Remark 7.1. If we take a set $\{X_i, i \in \mathbb{I}\}$ of objects of $\mathcal{C}$ and apply the same two definitions, we obtain **the product** $\prod_{i\in\mathbb{I}} X_i$ and **the coproduct** $\coprod_{i\in\mathbb{I}} X_i$

Exercise 7.6. Prove that the product of X and Y is the limit of a diagram with two vertices X and Y and no arrows. Prove that the coproduct of X and Y is the colimit of this diagram.

Exercise 7.7. Prove that the products in the category of sets, vector spaces and topological spaces are the usual products of sets, vector spaces and topological spaces.

Exercise 7.8. Prove that the coproduct in the category of sets and topological spaces is the disconnected union.

Exercise 7.9. Prove that a product $\prod_{i\in\mathbb{I}} X_i$ in the category of vector spaces is the usual product, and the coproduct $\coprod_{i\in\mathbb{I}} X_i$ is the direct sum.

Exercise 7.10. Prove that coproduct of $\mathbb{Z}$ with itself in the category of groups is the fundamental group of a graph with one vertex and two loops connecting this vertex with itself.

Exercise 7.11. Prove that coproduct of $\mathbb{Z}$ with itself n times in the category of groups is the fundamental group of a graph with one vertex and n loops connecting this vertex with itself.

Exercise 7.12. Prove that the coproduct in the category of rings with unity over $\mathbb{C}$ is the tensor product of rings over $\mathbb{C}$.

Exercise 7.13. Prove that the coproduct in the category of rings with unity is the tensor product of rings over $\mathbb{Z}$.