Home assignment 8: Exercises in divisibility

Definition 8.1. An element p of a ring R is called **irreducible** if for any decomposition $p = p_1 p_2$, either p_1 or p_2 is invertible, and **prime** if the ideal principal ideal (p) is prime We say that a ring R **admits the irreducible decomposition** if any element is a product of irreducible elements. R is called **factorial** if any element $r \in R$ admits the irreducible decomposition $r = \prod_i p_i^{\alpha_i}$, all p_i are prime, and this decomposition is unique up to invertible factors and permutation of p_i.

Exercise 8.1. Consider the ring $R := \mathbb{Z}[\sqrt{-p}]$, where $p \in \mathbb{Z}$ is a prime number. Prove that R admits the irreducible decomposition.

Exercise 8.2. Let $R := \mathbb{Z}[\sqrt{p}]$, where $p \in \mathbb{Z}$ is a prime number. Prove that R has no zero divisors.

Exercise 8.3. Consider the ring $R := \mathbb{Z}[\sqrt{p}]$, where $p \in \mathbb{Z}$ is a prime number. Prove that R admits the irreducible decomposition.

Exercise 8.4. Consider the ring $R := \mathbb{Z}[\sqrt{-3}]$. Prove that R is not factorial.

Exercise 8.5. Find whether the following numbers $p \in \mathbb{Z}$ are irreducible in the ring $\mathbb{Z}[\sqrt{-3}]$. Give the proof.

a. $p = 11$.

b. $p = 13$.

c. $p = 17$.

Exercise 8.6. Prove that the ideal $(2, 1 + \sqrt{-5})$ is not principal in the ring $\mathbb{Z}[\sqrt{-5}]$.

Exercise 8.7. Construct a ring A and an irreducible element $p \in A$ such that the principal ideal (p) is not prime.

Exercise 8.8. Consider a ring $R \subset \mathbb{C}$, which is discrete as a subset of $\mathbb{C}$. Prove that R admits the irreducible decomposition.

Exercise 8.9. Find a non-principal ideal in the ring $\mathbb{Z}[\sqrt{-p}]$, for some prime $p > 5$.