Commutative algebra, final exam

Rules: Every student gets 12 exercises, the final grade is determined by the score. The mark is C for score 30-49, B for 50-69, A for 70-89, A+ for the higher score.

1 Hilbert's Nullstellensatz and Zariski topology

Definition 1.1. Boolean ring is a ring such that all its elements are idempotents, that is, satisfy $a^2 = a$.

Exercise 1.1 (10 points). Prove that all prime ideals in a boolean ring are maximal, or find a counterexample.

Definition 1.2. Spectrum of a ring A is the set of all prime ideals of A. **Zariski topology** on the spectrum is defined by the base $\{U_f\}$ of open sets, enumerated by $f \in A$, defined as follows: a prime ideal $\mathfrak{p}$ belongs to U_f if $f \notin \mathfrak{p}$. **Maximal spectrum** is the set of maximal ideals with the same topology.

Exercise 1.2 (10 points). Prove that the spectrum of a boolean ring with Zariski topology is Hausdorff.

Exercise 1.3 (20 points). Let M be a compact, connected manifold of positive dimension, and $A = C(M)$ the ring of continuous functions on M. Prove that any prime ideal of A is maximal, or find a counterxample.

Exercise 1.4 (20 points). Let $P(x, y)$ be an irreducible polynomial, and $R := \frac{\mathbb{C}[x,y]}{(P)}$. Prove that all ideals in R are principal, or find a counterexample.

Exercise 1.5 (10 points). Let $A = \mathbb{R}[t]$, and X be its maximal spectrum. Prove that X with its Zariski topology is homeomorphic to the maximal spectrum of $\mathbb{C}[t]$.

2 Noetherian rings

Remark 2.1. In this section, the rings are not finitely-generated or Noetherian unless noted otherwise.

Definition 2.1. A module is **finitely representable** if it is a quotient of a free module by a finitely generated module.

Exercise 2.1 (10 points). Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of A-modules, and M_1, M_3 are finitely representable. Prove that M_2 is also finitely representable.

Definition 2.2. A prime ideal is called **minimal** if it does not contain any smaller prime ideals.

Exercise 2.2 (10 points). Let A be a finitely generated ring over $\mathbb{C}$. Prove that the number of minimal prime ideals in A is finite.

Exercise 2.3 (10 points). Let M be an A-module, where A is not necessarily Noetherian, and $N_1, N_2 \subset M$ submodules which satisfy $N_1 \cap N_2 = 0$. Assume that the modules M/N_1 and M/N_2 are Noetherian. Prove that M is Noetherian.

3 Group representations and categories

Exercise 3.1 (20 points). Let C be a smooth 1-dimensional affine variety over $\mathbb{C}$, equipped with an action of a finite group G. Prove that C/G is also smooth.

Exercise 3.2 (10 points). Prove that the category of finite groups is not equivalent to the category of finite abelian groups.

Exercise 3.3 (20 points). Let $\mathcal{C}_1$ be the category of finite-dimensional complex representations of the group $\mathbb{Z}/p\mathbb{Z}$, and $\mathcal{C}_2$ the category of all finite-dimensional complex representations of $\mathbb{Z}/q\mathbb{Z}$, where $p \neq q$ are prime numbers. Prove that the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are non-equivalent, or find a counterexample.

Exercise 3.4 (20 points). Let $\mathcal{C}_1$ be the category of finite-dimensional complex representations of the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{C}_2$ the category of all finite-dimensional complex representations of $\mathbb{Z}/4\mathbb{Z}$. Are the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ equivalent?

Exercise 3.5 (20 points). Let G be the group of symmetries of a square, acting on $\mathbb{R}^2$. We induce the action of G on $\mathbb{C}^2 = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2$ in a natural way. Prove that $\mathbb{C}^2/G$ is smooth.

4 Flat modules

Definition 4.1. A module M over a ring R is called **flat** if the functor $X \longrightarrow X \otimes_R M$ is exact.

Definition 4.2. Let A be a ring without zero divisors. **Torsion** in an A-module M is the kernel of a natural map $M \longrightarrow M \otimes_A k(A)$.

Exercise 4.1 (10 points). Let M be a flat R-module, where R is a ring without zero divisors. Prove that M is torsion-free.

Exercise 4.2 (20 points). Let $M_1 \subset M_2 \subset ...$ be a sequence of embedded R-modules. Assume that all M_i are flat. Prove that $\bigcup M_i$ is also flat.

Exercise 4.3 (20 points). Prove that a finitely generated module over a Noetherian local ring is flat if and only if it is free.

Exercise 4.4 (20 points). Let A be a ring without zero divisors M be an A-module which is flat over all local rings $A_s \supset A$ with the same fraction field. Prove that M is flat.

Exercise 4.5 (30 points). Prove that any finitely generated module over a Boolean ring is flat.

5 Tensor product

Exercise 5.1 (10 points). Let R_1, R_2 be rings over $\mathbb{C}$ such that $R_1 \otimes_{\mathbb{C}} R_2$ is finitely generated. Prove that R_1, R_2 are finitely generated, or find a counterexample.

Definition 5.1. An R-module M is called **invertible** if $M \otimes_R M^* \cong R$, where $M^* = \operatorname{Hom}_R(M, R)$.

Exercise 5.2 (10 points). Prove that each finitely generated, invertible R-module admits a monomorphism to a free R-module.

Exercise 5.3 (10 points). Let P be the set of isomorphism classes of invertible R-modules. Prove that tensor product defines a group structire on P (this group is called **Picard group** of P). Find P for a ring $\mathbb{C}[t]$.

Exercise 5.4 (10 points). Let M, N be finitely generated modules over the ring $R = \mathbb{C}[[t_1, ..., t_n]]$ of power series, and $M \otimes_R N = 0$. Prove that either $M = 0$ or $N = 0$.

6 Normality, irreducibility, smoothness

Exercise 6.1 (10 points). Let A be the ring of complex-analytic functions on $\mathbb{C}$, and $k(A)$ its fraction field. Prove that the transcendence degree of A over $\mathbb{C}$ is infinite.

Exercise 6.2 (10 points). Let A be an integrally closed ring, G a finite group acting on A by automorphisms, and A^G the ring of invariants. Prove that A^G is integrally closed.

Exercise 6.3 (10 points). Prove that the ring $\mathbb{C}[x, y, z]/(y^2 - xz)$ is integrally closed.

Exercise 6.4 (20 points). Let $\zeta_n \in \mathbb{C}$ be a primitive root of unity of degree n. Define the action of the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ with coordinates x, y in such a way that the generator t maps x to $\zeta_n x$ and y to $\zeta_n^{-1} y$. Prove that the quotient $\mathbb{C}^2/G$ is singular.

Exercise 6.5 (20 points). Let $A \subset \mathbb{C}^n$ be a subvariety given by equations $z_1^{p_1} = z_2^{p_2} = ...z_n^{p_n}$. Suppose that all p_i are different primes. Prove that A is irreducible.

Exercise 6.6 (10 points). Let $P(x, y, z)$ and $Q(x, y, z) \in \mathbb{C}[x, y, z]$ be irreducible, coprime polynomials, and X a variety defined by $P = Q = 0$. Prove that X is irreducible, or find a counterexample.

Exercise 6.7 (20 points). Let $A \subset \mathbb{C}^n$ be a subvariety given by equations $z_1^{p_1} = z_2^{p_2} = ...z_n^{p_n}$. Suppose that all p_i are different primes. Prove that A is irreducible.

Exercise 6.8 (20 points). Let $P(x, y, z)$ and $Q(x, y, z) \in \mathbb{C}[x, y, z]$ be irreducible, coprime polynomials, and X a variety defined by $P = Q = 0$. Prove that X is irreducible, or find a counterexample.

Exercise 6.9 (10 points). Let $S \subset \mathbb{C}^2$ be a smooth hypersurface defined by an irreducible quadratic equation. Prove that S is isomorphic to $\mathbb{C}$ or to $\mathbb{C}\backslash\{0\}$.

Exercise 6.10 (20 points). Let $S \subset \mathbb{C}^n$ be a smooth hypersurface given by a quadratic equation. Prove that S is normal.