Commutative algebra

lecture 3: Strong Nullstellensatz

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IMPA, sala 2

January 7, 2022

REMINDER: Affine varieties and finitely generated rings

DEFINITION: Category of affine varieties over $\mathbb{C}$: its objects are algebraic subsets in $\mathbb{C}^n$, morphisms – polynomial maps.

DEFINITION: Finitely generated ring over $\mathbb{C}$ is a quotient of $\mathbb{C}[t_1, ..., t_n]$ by an ideal.

DEFINITION: Let R be a ring. An element $x \in R$ is called **nilpotent** if $x^n = 0$ for some $n \in \mathbb{Z}^{>0}$.

Theorem 1: Let $\mathcal{C}_R$ be a category of finitely generated rings over $\mathbb{C}$ without non-zero nilpotents and $\mathcal{Aff}$ – category of affine varieties. Consider the functor $\Phi: \mathcal{Aff} \longrightarrow \mathcal{C}_R^{op}$ mapping an algebraic variety X to the ring of polynomial functions on X. **Then Φ is an equivalence of categories.**

Proof: Later in this lecture.

Strong Nullstellensatz

DEFINITION: Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal. Denote the **set of common zeros** for I by $V(I)$, with

$$V(I) = \{(z_1, ..., z_n) \in \mathbb{C}^n \ \mid \ f(z_1, ..., z_n) = 0 \forall f \in I\}.$$

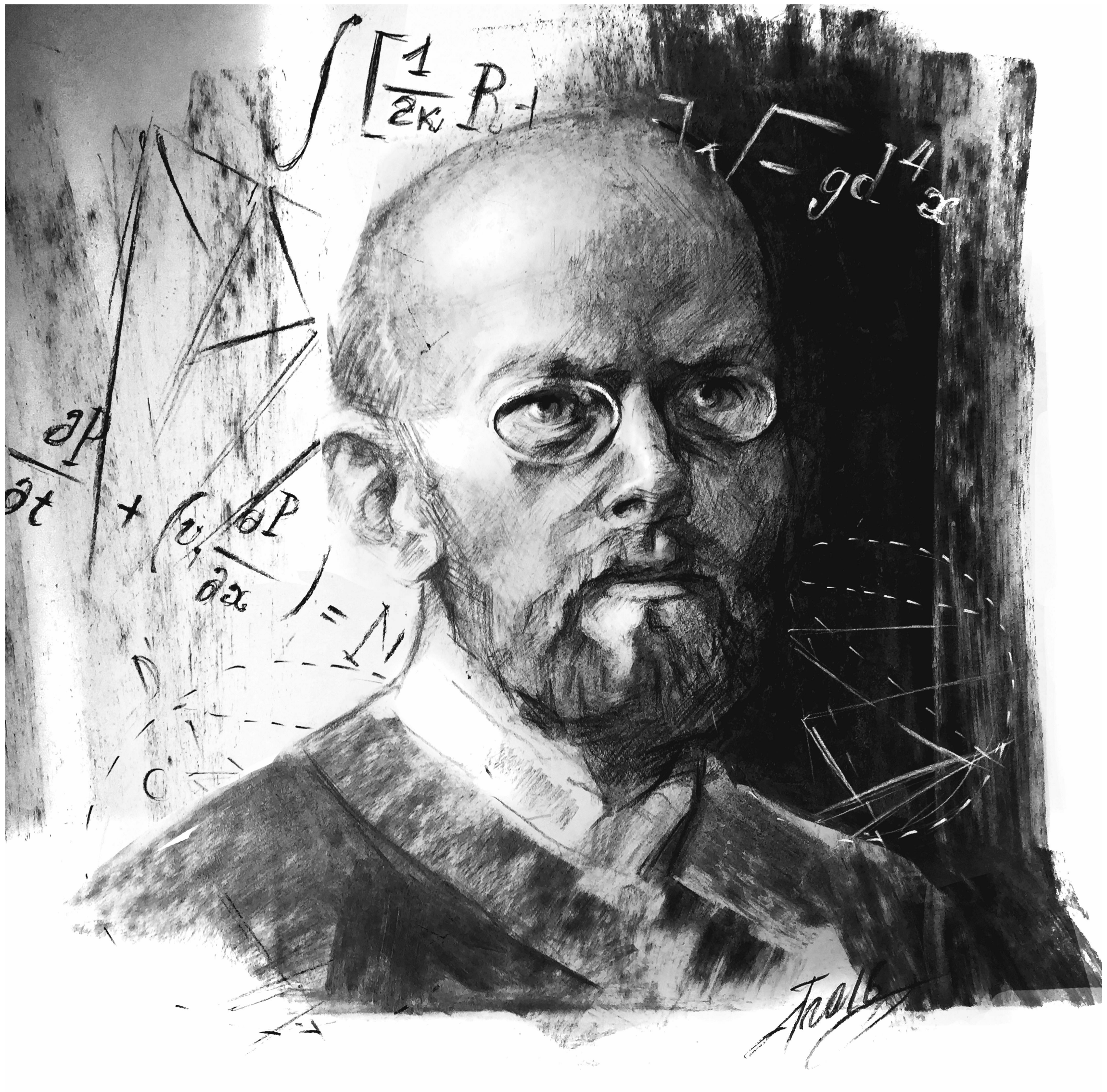
For $Z \subset \mathbb{C}^n$ an algebraic subset, denote by $\operatorname{Ann}(A)$ the set of all polynomials $P(t_1, ..., t_n)$ vanishing in Z.

THEOREM: **(strong Nullstellensatz).** For any ideal $I \subset \mathbb{C}[t_1, ..., t_n]$ such that $\mathbb{C}[t_1, ..., t_n]/I$ has no nilpotents, **one has** $\operatorname{Ann}(V(I)) = I$**.**

Proof: Later in this lecture.

REMARK: "Weak Nullstellensatz" claims that $V(I)$ is never empty for any ideal I; "Strong Nullstellensatz" claims that I is uniquely determined by $V(I)$ when R/I has no nilpotents.

Strong Nullstellensatz and equivalence of categories



David Hilbert (Anna Gorban, 2018)

Strong Nullstellensatz and equivalence of categories

THEOREM: (strong Nullstellensatz). For any ideal $I \subset \mathbb{C}[t_1, ..., t_n]$ such that $\mathbb{C}[t_1, ..., t_n]/I$ has no nilpotents, **one has** $\operatorname{Ann}(V(I)) = I$**.**

Now we deduce Theorem 1 from Strong Nullstellensatz. This would require us to construct a functor $\Psi : \mathcal{C}_R^{op} \longrightarrow \mathcal{Aff}$. Since any object $R \in \mathcal{Ob}(\mathcal{C}_R)$ is given as $R = \mathbb{C}[t_1, ..., t_n]/I$, we define Ψ as $\Psi(R) := V(I)$; the functor $\Phi : \mathcal{Aff} \longrightarrow \mathcal{C}_R^{op}$ was defined as $Z \longrightarrow \operatorname{Ann}(Z)$.

Strong Nullstellensatz gives $\operatorname{Ann}(V(I)) = I$, hence $\Phi(\Psi(R)) = R$ **for any finitely generated ring.** It remains to prove $V(\operatorname{Ann}(Z)) = Z$.

Clearly, $V(\operatorname{Ann}(Z)) \supset Z$: any point $z \in Z$ belongs to the set of common zeros of $\operatorname{Ann}(Z)$. On the other hand, Z is a set of common zeros of a system $\mathcal{P}$ of polynomial equations, giving $Z = V(\mathcal{P}) \supset V(\operatorname{Ann}(Z))$.

Localization

DEFINITION: Localization $R(F))$ of a ring R with respect to $F \in R$ is a ring $R[F^{-1}]$, which is formally generated by the elements of form a/F^n and relations $a/F^n \cdot b/F^m = ab/F^{n+m}$, $a/F^n + b/F^m = \frac{aF^m + bF^n}{F^{n+m}}$, and $aF^k/F^{k+n} = a/F^n$.

REMARK: Clearly, $R(F) = R[t]/(tF-1)$.
EXAMPLE: $\mathbb{Z}[2^{-1}]$, the ring of rational numbers with denominators 2^k.
EXAMPLE: $\mathbb{C}[t, t^{-1}]$, the ring of Laurent polynomials.
EXERCISE: Let R be a finitely generated ring over a field k. **Prove that $R[F^{-1}]$ is a finitely generated ring over k.**

DEFINITION: $a \in R$ is called **nilpotent** if $a^n = 0$ for some $n > 0$.

CLAIM 1: Suppose that $R[F^{-1}] = 0$, where $F \in R$. **Then F is nilpotent.**

Proof. Step 1: $R(F) = R[t]/(tF-1)$. **Therefore, $1 = 0$ implies $1 = (Ft-1)P$, for some $P \in R[t]$.**

Step 2: Let $P(t) = \sum a_i t^i$, where $a_i \in R$. **Then $1 = (Ft-1)P$ implies $a_i = a_{i-1}F$ for all $i > 0$, and $a_0 = 1$.**

Step 3: This gives $P = \sum F^i t^i$, and $F^{n+1} = 0$. ■

Spectrum and localization

DEFINITION: Spectrum of a ring R is the set $\operatorname{Spec} R$ if its prime ideals.

EXERCISE: Let $R \xrightarrow{\varphi} R_1$ be a ring homomorphism. Prove that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal, for any $\mathfrak{p} \in \operatorname{Spec} R_1$.

PROPOSITION: In other words, any morphism $R \longrightarrow R_1$ **gives an injective map of spectra** $\operatorname{Spec} R[f^{-1}] \hookrightarrow \operatorname{Spec} R$.

Proof: Suppose that $\mathfrak{p}_f, \mathfrak{q}_f \in \operatorname{Spec} R(f)$, and $\mathfrak{p} = \mathfrak{q}$ are their images in $\operatorname{Spec} R$. Then for each $p \in \mathfrak{p}_f$, we have $f^N p \in \mathfrak{q} \subset \mathfrak{q}_f$; since $\mathfrak{q}$ is prime, this implies that $p \in \mathfrak{q}$. ■

DEFINITION: Nilradical of a ring R is the set $\operatorname{Nil}(R)$ of all nilpotent elements of R.

THEOREM: Interesection P of all prime ideals of R is equal to $\operatorname{Nil}(R)$.

Proof: Clearly, $P \supset \operatorname{Nil}(R)$. Assume that, conversely, $x \notin \operatorname{Nil}(R)$. Then $R[x^{-1}] \neq 0$, **hence $R[x^{-1}]$ contains a prime ideal** (the maximal one), and its image in $\operatorname{Spec} R$ does not containt x. ■

Rabinowitz trick

DEFINITION: Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal. Recall that the set of common zeros of I is denoted by $V(I)$ **("vanishing set", "null-set", "zero set"),** and the set of all polynomials vanishing in $Z \subset \mathbb{C}^n$ is denoted $\operatorname{Ann}(Z)$ **("annihilator").**

Theorem 1: Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal, and f a polynomial function, vanishing on $V(I)$. **Then $f^N \in I$ for some $N \in \mathbb{Z}^{>0}$.**

Proof. Step 1: Consider an ideal $I_1 \subset \mathbb{C}[t_1, ..., t_{n+1}]$ generated by $I \subset \mathbb{C}[t_1, ..., t_n]$ and $ft_{n+1} - 1$. **Since the submodule of R generated by $\langle ft_{n+1} - 1, I\rangle$ has no common zeros, I_1 contains 1** by (weak) Nullstellensatz.

Step 2: Let $R := \mathbb{C}[t_1, ..., t_n]/I$. Consider the surjective map $\zeta : \mathbb{C}[t_1, ..., t_{n+1}] \longrightarrow R[f^{-1}]$ taking $t_1, ..., t_n$ to their images in R and mapping t_{n+1} to f^{-1}. Since $\zeta(I_1) = 0$, and $1 \in I_1$, one has $1 = 0$ in $R[f^{-1}]$, giving $R[f^{-1}] = 0$. **By Claim 1, f is nilpotent in R.** ∎

COROLLARY: (Strong Nullstellensatz)
Suppose that $R := \mathbb{C}[t_1, ..., t_n]/I$ is a ring without nilpotents. **Then $I = \operatorname{Ann}(V_I)$).**

Proof: If $a \in \operatorname{Ann}(V_I)$), then $a^n \in I$ by Theorem 1. ∎

Zum Hilbertschen Nullstellensatz

The first time Hilbert Nullstellensatz was called by this name.

Zum Hilbertschen Nullstellensatz.

Von

J. L. Rabinowitsch in Moskau.

Satz. *Verschwindet das Polynom $f(x_1, x_2, \ldots, x_n)$ in allen Nullstellen — im algebraisch abgeschlossenen Körper — eines Polynomideals $\mathfrak{a}$, so gibt es eine Potenz f^ϱ von f, die zu $\mathfrak{a}$ gehört.*

Beweis. Es sei $\mathfrak{a} = (f_1, f_2, \ldots, f_r)$, wo f_i die Variablen $x_1, \ldots, x_n$ enthalten. x_0 sei eine Hilfsvariable. Wir bilden das Ideal $\bar{\mathfrak{a}} = (f_1, f_2, \ldots, f_r, x_0 f - 1)$. Da der Voraussetzung nach $f = 0$ ist, sobald alle f_i verschwinden, so hat das Ideal $\bar{\mathfrak{a}}$ keine Nullstellen.

Folglich muß $\bar{\mathfrak{a}}$ mit dem Einheitsideal zusammenfallen. (Vgl. etwa bei K. Hentzelt, „Eigentliche Eliminationstheorie", § 6, Math. Annalen 88 [1]).) Ist also $1 = \sum_{i=1}^{i=r} F_i(x_0, x_1, \ldots, x_n) f_i + F_0 \cdot (x_0 f - 1)$ und setzen wir in dieser Identität $x_0 = \frac{1}{f}$, so ergibt sich:

$$1 = \sum_{i=1}^{i=r} F_i\left(\frac{1}{f}, x_1, \ldots, x_n\right) f_i = \frac{\sum_{i=1}^{i=r} \bar{F}_i f_i}{f^\varrho}.$$

Folglich ist $f^\varrho = 0\,(\mathfrak{a})$, w. z. b. w.

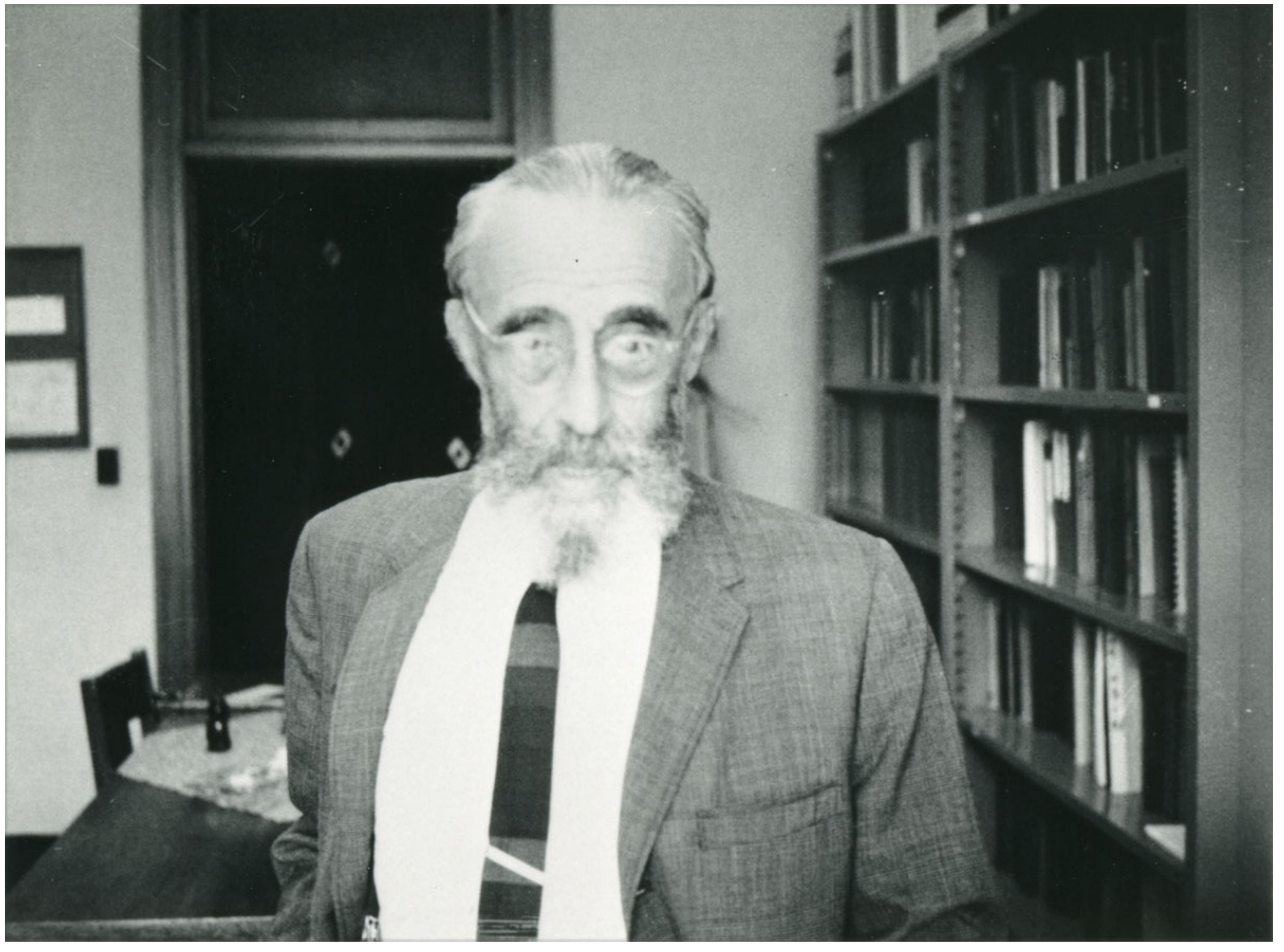
[1]) Folgt auch schon aus der Kroneckerschen Eliminationstheorie.

(Eingegangen am 8. 5. 1929.)

J.L. Rabinowitsch, *Zum Hilbertschen Nullstellensatz,*
Mathematische Annalen (1930), 102, p. 520-520

George Rainich

George Yuri Rainich (Rabinovich), 1886-1968.



Photograph by Paul R. Halmosh, 1964, Ann Arbor.

Hilbert Nullstellensatz as equivalence of categories

Theorem 1: Let $\mathcal{C}_R$ be a category of finitely generated rings over $\mathbb{C}$ without non-zero nilpotents and $\mathcal{Aff}$ – category of affine varieties. Consider the functor $\Phi: \mathcal{Aff} \longrightarrow \mathcal{C}_R^{op}$ mapping an algebraic variety X to the ring of polynomial functions on X. **Then Φ is an equivalence of categories.**

Proof: Let $R = \mathbb{C}[t_1, ..., t_n]/I$. The functor $\Psi: \mathcal{C}_R^{op} \longrightarrow \mathcal{Aff}$ takes R to V_I (the common zeros set of I). Then $\Psi \circ \Phi$ takes R to the ring of polynomial functions on V_I, which is equal to $\frac{\mathbb{C}[t_1,...,t_n]}{\operatorname{Ann}(V_I)}$. Since $\operatorname{Ann}(V_I) = I$, this functor takes R to itself.

For another direction, $\Phi \circ \Psi$ takes an algebraic set A to the common zeros of the ideal $\operatorname{Ann}(A)$, which is the same as A, because $A = V_{\operatorname{Ann}(A)}$, by definition of an algebraic set. ∎