Commutative Algebra

lecture 4: Noetherian rings and irreducible affine varieties

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Irreducible varietiees

DEFINITION: An affine variety A is called **reducible** if it can be expressed as a union $A = A_1 \cup A_2$ of affine varieties, such that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$. If such a decomposition is impossible, A is called **irreducible**.

CLAIM: An affine variety A is **irreducible** if and only if its ring of polynomial functions $\mathcal{O}_A$ **has no zero divizors.**

Proof: If $A = A_1 \cup A_2$ is a decomposition of A into a non-trivial union of subvarieties, choose a non-zero function $f \in \mathcal{O}_A$ vanishing at A_1 and g vanishing at A_2. The product of these non-zero functions vanishes in $A = A_1 \cup A_2$, **hence $fg = 0$ in $\mathcal{O}_A$.** Conversely, **if $fg = 0$, we decompose $A = V_f \cup V_g$.** ■

Noetherian rings and irreducible components

DEFINITION: A ring is called **Noetherian** if any increasing chain of ideals stabilizes: for any chain $I_1 \subset I_2 \subset I_3 \subset ...$ one has $I_n = I_{n+1} = I_{n+2} = ...$

THEOREM: (Hilbert basis theorem)
Any finitely generated ring is Noetherian.
Proof: *Later today.*

DEFINITION: An irreducible component of an algebraic set A is an irreducible algebraic subset $A' \subset A$ such that $A = A' \cup A''$, and $A' \not\subset A''$.

Remark 1: Let $A_1 \supset A_2 \supset ... \supset A_n \supset ...$ be a decreasing chain of algebraic subsets in an algebraic variety. **Then the corresponding ideals form an increasing chain of ideals:** $\operatorname{Ann}(A_1) \subset \operatorname{Ann}(A_2) \subset \operatorname{Ann}(A_3) \subset ...$

THEOREM: Let A be an affine variety, and $\mathcal{O}_A$ its ring of polynomial functions. Assume that $\mathcal{O}_A$ is Noetherian. Then A **is a union of its irreducible components, which are finitely many.**

Proof: See the next slide ■

Remark 2: From the noetherianity and Remark 1 it follows that A **cannot contain a strictly decreasing infinite chain of algebraic subvarieties.**

Noetherian rings and irreducible components (2)

THEOREM: Let A be an affine variety, and $\mathcal{O}_A$ its ring of polynomial functions. Assume that $\mathcal{O}_A$ is Noetherian. Then **A is a union of its irreducible components, which are finitely many.**

Proof. Step1: Each point $a \in A$ belongs to a certain irreducible component. Indeed, suppose that such a component does not exist. Then for each decomposition $A = A_1 \cup A_2$ of A onto algebraic sets, the set A_i containing a can be split non-trivially onto a union of algebraic sets, the component containing a can also be split, and so on, *ad infinitum*. This gives a strictly decreasing infinite sequence, a contradiction (Remark 2).

Step 2: We proved existence of an irreducible decomposition, and **it remains only to show that number of irreducible components of A is finite.** Let $A = \bigcup A_i$ be an irreducible decomposition.

Step 3: Let **algebraic closure** of a set $X \subset \mathbb{C}^n$ be the intersection of all algebraic subsets of $\mathbb{C}^n$ containing X. An intersection of algebraic sets is algebraic, because it is defined by the ideal generated by the union of their ideals. Since $A = A_i \cup \bigcup_{j\neq i} A_j$, the algebraic closure B_i of $A \backslash A_i$ does not contain A_i. and **the sequence $B_1 \supset B_1 \cap B_2 \supset B_1 \cap B_2 \cap B_3 \subset ...$ decreases strictly, unless there are only finitely many irreducible components.** Applying Remark 2 again, we obtain that the number of B_i is finite. ■

Noetherian rings

DEFINITION: A finitely generated ring is a quotient of a polynomial ring.

THEOREM: (Hilbert's Basis Theorem)
Any finitely generated ring over a field is Noetherian.

Proof: Later in this lecture.

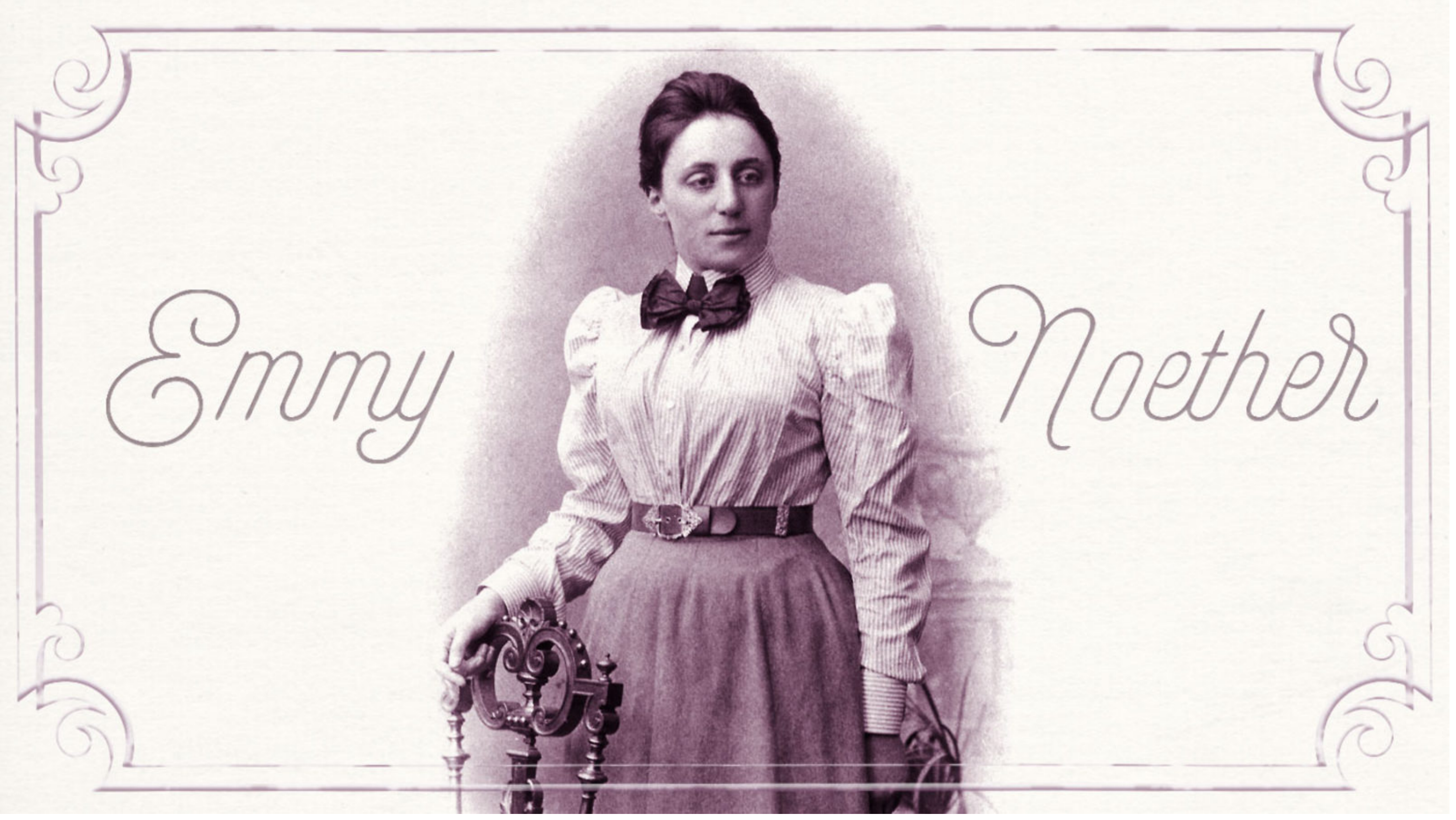
COROLLARY: For any affine manifold, **its ring of functions is Noetherian,** hence the the irreducible decomposition exists and is finite.

REMARK: It suffices to prove Hilbert's Basis Theorem for the ring of polynomials. Indeed, any finitely generated ring is a quotient of the polynomial ring, but **the set of ideals of the quotient ring A/I is injectively mapped to the set of ideals of R.**

REMARK: Therefore, **Hilbert's Basis Theorem would follow if we prove that $R[t]$ is Noetherian for any Noetherian ring R.**

EXERCISE: Find an example of a ring which is not Noetherian.

Emmy Noether



Amalie Emmy Noether (1882-1935).

Emanuel Lasker

Emanuel Lasker (1868-1941).

Finitely generated ideals

DEFINITION: **Finitely generated ideal** in a ring is an ideal $\langle a_1, ..., a_n\rangle$ of sums $\sum b_i a_i$, where $\{a_i\}$ is a fixed finite set of elements of R, called **generators** of R.

LEMMA: Let $I \subset R$ be a finitely generated ideal, and $I_0 \subset I_1 \subset I_2 \subset ...$ an increasing chain of ideals, such that $\bigcup_n I_n = I$. **Then this chain stabilises**.

Proof: Let $I = \langle a_1, ..., a_n\rangle$, and I_N be an ideal in the chain $I_0 \subset I_1 \subset I_2 \subset ...$ which contains all a_i. Then $I_N = I$. ■

CLAIM: A ring R is Noetherian if and only if all its ideals are finitely generated.

Proof: For any chain of ideals $I_0 \subset I_1 \subset I_2 \subset ...$, **finite generatedness of $I = \bigcup I_i$ guarantees stabilization of this chain,** as follows from Lemma above.

Conversely, if R is Noetherian, and I any ideal, take $I_0 = 0$ and let $I_k \subset I$ be obtained by adding to I_{k-1} an element of I not containing in I_{k-1}. **Since the chain $\{I_k\}$ stabilizes, I is finitely generated.** ■

Noetherian modules

DEFINITION: A module over a ring R is a vector space M equipped with an algebra homomorphism $R \longrightarrow \mathrm{End}(M)$.

EXAMPLE: A subspace $I \subset R$ in a ring **is an ideal if and only if I is an R-submodule of R,** considered as an R-module.

DEFINITION: A module M over R is called **Noetherian** if any increasing chain of submodules of M stabilizes.

REMARK: Any submodules and quotient modules of a Noetherian R-module are again Noetherian.

Finitely generated R-modules

DEFINITION: An R-module is called **finitely generated** if it is a quotient of **a free module** R^n by its submodule.

EXERCISE: Show that **a module M is Noetherian iff any $M' \subset M$ is finitely generated.** Use this to prove that **finite direct sums of Noetherian modules are Noetherian.**

LEMMA: A ring R is Noetherian if and only if it is Noetherian as an R-module.

Proof: Ideals in R is the same as R-submodules of R, stabilization of a chain of R-submodules in R is literally the same as stabilization of a chain of ideals in R. ■

REMARK: Let M be a module over $R[t]$ which is Noetherian as an R-module, **Then it is Noetherian as $R[t]$-module.** ■

COROLLARY: If R is Noetherian, then $R[t]/(t^N) = R^N$ is a Noetherian R-module. Therefore, **the ring $R[t]/(t^N)$ is Noetherian.** ■

An ideal generated by leading terms

Proposition 1: Let $J \subset R[t]$ be an ideal, and $Q_1(t), ..., Q_n(t) \in J$ be polynomials of degree k. Let $J_0 \subset R$ be an ideal generated by the leading terms of $Q_i(t)$. Suppose that all leading terms of all $P(t) \in J$ belong to J_0. Then, **for each $P(t) \in J$, there is an element $P_1(t) \in J$, $\deg P_1(t) < k$, such that $P(t) = P_1(t) \mod J_Q$,** where $J_Q = (Q_1(t), ..., Q_n(t))$ is the ideal generated by all $Q_i(t)$.

Proof. Step1: Let $P(t), Q(t)$ be polynomials in $R[t]$, $n = \deg P > k = \deg Q$, p, q their leading coefficients. Suppose that $p = q$. **Then $P(t) = Q(t)t^{n-m} + P_1(t)$, where $\deg P_1 < \deg P$.**

Step 2: For each $P(t) \in J$ of degree $n > k$, there exists a polynomial $Q(t) = t^{n-k} \sum u_i Q_i(t) \in J_Q$, $u_i \in R$, with the same degree and the leading term. **Then $\deg(P(t) - Q(t)) < \deg P(t)$.** Using induction by $\deg P$, we may assume that $P(t) - Q(t) = P_1(t) \mod J_Q$, where $\deg P_1(t) < k$, hence $P(t) = P_1(t) \mod J_Q$. ■

Proof of Hilbert's basis theorem

PROPOSITION: Let R be a Noetherian ring. **Then the polynomial ring $R[t]$ is also Noetherian.**

Proof. Step1: Let $I \subset R[t]$ be an ideal. We need to show that it is finitely generated. Consider the ideal $I_0 \subset R$ generated by all leading coefficients of all $P(t) \in I$. Since R is Noetherian, I_0 is finitely generated: $I_0 = \langle a_1, ..., a_n \rangle$, **where all a_i are leading coefficients of the finite set of polynomials $Q_i(t) \in I$.**

Step 2: Let N be the maximum of all degrees of $Q_i(t)$, and $I_Q \subset R[t]$ the ideal generated by all $Q_i(t)$. By Proposition 1, **for any $P(t) \in I$, one has $P(t) = P_1(t) \mod J_Q$, where $\deg P_1(t) < N$.**

Step 3: We have shown that the natural map $I \cap R\langle 1, t, t^2, ..., t^{N-1} \rangle \longrightarrow I/I_Q$ is surjective. However, $I \cap R\langle 1, t, t^2, ..., t^{N-1} \rangle \subset R^N$ is Noetherian as an R-module, hence $M := I/I_Q$ is finitely generated as an R-module.

Step 4: Pick a set of polynomials $U_1(t), ..., U_m(t) \in I$, generating M. **Then $\{Q_i(t), U_i(t)\}$ generate I.** ■

Smooth points

DEFINITION: Let $A \subset \mathbb{C}^n$ is an algebraic subset. A point $a \in A$ is called **smooth**, if there exists a neighbourhood U of $a \in \mathbb{C}^n$ (in the usual topology) such that $A \cap U$ is a smooth $2k$-dimensional real submanifold. A point is called **singular** if such diffeomorphism does not exist. A variety is called **smooth** if it has no singularities, and **singular** otherwise.

PROPOSITION: For any algebraic variety A and any smooth point $a \in A$, a diffeomorphism between a neighbourhood of a and an open ball **can be chosen polynomial**.

Proof. Step1: If $a \in A \subset \mathbb{C}^n$ is a smooth point of a k-dimensional embedded manifold, **there exists k complex linear functions on $\mathbb{C}^n$ such that their differentials are independent in the tangent space $T_a A$.**

Step 2: By inverse function theorem, these functions **define a diffeomorphism from a neighbourhood of A to an open subset of $\mathbb{C}^k$.** ■

Analytic functions

REMARK: In the coordinates defined by linear functions **all regular (polynomial) functions on A are analytic** (equal to the sum of their Taylor series).

In other polynomial coordinate systems, the Taylor series may be no longer finite, but **the regular functions remain analytic,** because the inverse function theorem remains true in analytic category.

REMARK: In fact, **all complex differentiable functions are analytic** (Cauchy), and the regular functions are clearly complex differentiable.

Irreducibility for smooth varieties

CLAIM: Let M be an algebraic variety which is smooth and connected. **Then it is irreducible.**

Proof: ("Analytic continuation principle"). Step 1
Let f, g be non-zero polynomial functions, $fg = 0$. Decomposing f, g onto Taylor series around $m \in M$, **we obtain that the Taylor series for f or for g vanish.** Suppose it is f which has vanishing Taylor series, and let $U \subset M$ be the set where all derivatives of f vanish.

Step 2: Since U is an intersection of closed sets $\{x \in M \mid f^{(i)}(x) = 0\}$, it is closed. However, an analytic function which has vanishing Taylor series in x has to vanish in a neighbourhood of x, hence U is also open. **An open and closed subset of M is M or $\emptyset$, because M is connected.** ■

Irreducibility for smooth varieties (2)

COROLLARY: Let A be an affine variety such that its set A_0 of smooth points is dense in A and connected. **Then A is irreducible.**

Proof: If f and g are non-zero function such that $fg = 0$, the ring of polynomial functions on A_0 contains zero divizors. However, **on a smooth, connected complex manifold the ring of polynomial functions has no zero divisors by analytic continuation principle.** ■

REMARK: Converse is also true: **an algebraic variety over $\mathbb{C}$ is irreducible if and only if the set of its smooth points is connected.** This is a complicated result.

EXERCISE: Let $X \longrightarrow Y$ be a morphism of affine manifols, where X is irreducible, and its image in Y is dense. **Prove that Y is also irreducible.**