Commutative Algebra

lecture 9: Yoneda lemma and the fibered product

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Natural transformation of functors

DEFINITION: Let $F, G: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be functors on categories. **A natural transformation of functors** is a morphism $\Psi_X: F(X) \longrightarrow G(X)$ such that for any $\varphi \in \mathcal{M}or(X, Y)$, one has $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$.

REMARK: The condition $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$ **is expressed by a commutative diagram**

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ \Psi_X \downarrow & & \downarrow \Psi_Y \\ G(X) & \xrightarrow{G(\varphi)} & G(Y) \end{array}$$

REMARK: Equivalence of functors **is a special case of a natural transformation of functors.**

Representable functors and natural transformations

DEFINITION: Consider the functor $h_A : \mathcal{C} \longrightarrow \mathcal{S}ets$ taking $X \in \mathcal{O}b(\mathcal{C})$ to $\mathcal{M}or(A, X)$. We say that h_A **is represented by an object $A \in \mathcal{O}b(\mathcal{C})$.**

CLAIM: Let $\Phi : h_A \longrightarrow F$ be a natural transformation of functors from $\mathcal{C}$ to sets. **Then Φ is uniquely determined by the element $\Phi(\mathrm{Id}_A) \in F(A)$.**

Proof: For any $\lambda \in \mathcal{M}or(A, B)$, we have a commutative diagram

$$\begin{tikzcd} h_A(A) = \mathcal{M}or(A,A) \arrow[r, "f \mapsto f\circ\lambda"] \arrow[d, "\Phi_A"'] & h_A(B) = \mathcal{M}or(A,B) \arrow[d, "\Phi_B"] \\ F(A) \arrow[r, "F(\lambda)"] & F(B) \end{tikzcd}$$

Suppose that the top arrow takes Id_A to λ. Commutativity of this diagram implies that $\Phi_B(\lambda) = F(\lambda)(\Phi_A(\mathrm{Id}_A))$, hence **the map $\Phi_B : \mathcal{M}or(A, B) \longrightarrow F(B)$ is uniquely determined by $\Phi_A(\mathrm{Id}_A) \in \mathcal{M}or(A, A)$.** ∎

Yoneda lemma

This brings the following useful result.

THEOREM: (Yoneda lemma)
Let $\mathcal{C}$ be a category, $A \in \mathcal{O}b(\mathcal{C})$, and $h_A : \mathcal{C} \longrightarrow \mathcal{S}ets$ the functor represented by A. Consider a functor $F : \mathcal{C} \longrightarrow \mathcal{S}ets$. Then **the set of natural transformations $h_A \longrightarrow F$ is in bijective correspondence with $F(A)$.** ∎

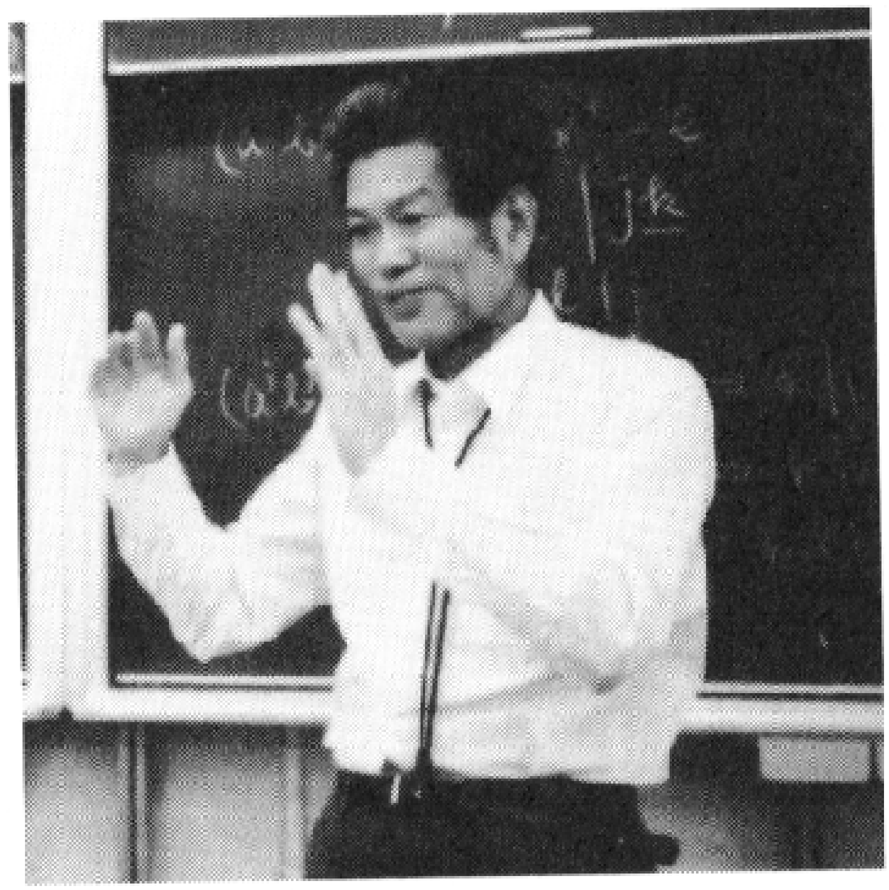
The functors $F : \mathcal{C} \longrightarrow \mathcal{S}ets$ form a category. Objects of this category are functors $F : \mathcal{C} \longrightarrow \mathcal{S}ets$, morphisms are natural transforms. Yoneda lemma **immediately implies that $\mathcal{M}or(h_A, h_B) = \mathcal{M}or(B, A)$.** We obtained the following statement.

CLAIM: Let $\mathcal{C}$ be a category, and $\mathcal{F}$ the category of representable functors $F : \mathcal{C} \longrightarrow \mathcal{S}ets$. **Then the contravariant functor $A \longrightarrow h_A$ defines an equivalence of categories $\mathcal{C} \longrightarrow \mathcal{F}^\circ$.** ∎

REMARK: In particular, **an object A of category $\mathcal{C}$ representing a given functor $h_A : \mathcal{C} \longrightarrow \mathcal{S}ets$ is uniquely up to an isomorphism determined by this functor**.

Yoneda lemma (2)

REMARK: The same is true for contravariant functors: **a category $\mathcal{C}$ is equivalent to the category $\mathcal{G}$ of contravariant functors $\mathcal{C}^\circ \longrightarrow \mathcal{S}ets$ represented by $h^\circ_A(X) = \mathcal{M}or(X, A)$.**



米田信夫

Yoneda Nobuo, (1930-1996)

Initial and terminal objects

DEFINITION: Fix a directed graph (graph with arrows). Suppose that for each vertex of this graph we have an object of category $\mathcal{C}$, and each arrow corresponds to a morphism of the associated objects. These data is called **a diagram** in the category $\mathcal{C}$. It is called **commutative** if whenever there exist two ways of going from one vertex to another along directed arrows, the compositions of the corresponding arrows are equal.

DEFINITION: **An initial object** of a category is an object $I \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(I, X)$ is always a set of one element. **A terminal object** is $T \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(X, T)$ is always a set of one element.

EXERCISE: Prove that **the initial and the terminal object is unique.**

Limits and colimits of diagrams

DEFINITION: Let $S = \{X_i, \varphi_{ij}\}$ be a commutative diagram in $\mathcal{C}$, and $\vec{\mathcal{C}}_S$ be a category of pairs (object X in $\mathcal{C}$, morphisms $\psi_i : X \longrightarrow X_i$, defined for all X_i) making the diagram formed by $(X, X_i, \psi_i, \varphi_{ij})$ commutative.

$$\begin{tikzcd}
X \arrow[dr, "\psi_1"] \arrow[drr, "\psi_3"] \arrow[ddr, "\psi_2"'] & & \\
& X_1 \arrow[r, "\varphi_{12}"] \arrow[d, "\varphi_{13}"'] & X_2 \arrow[dl, "\varphi_{23}"] \\
& X_3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "\Psi"] \arrow[drr, "\psi'_1"] \arrow[drrr, "\psi'_2"] \arrow[ddrr, "\psi'_3"'] & X \arrow[dr, "\psi_1"] \arrow[drr, "\psi_2"] \arrow[ddr, "\psi_3"'] & & \\
& & X_1 \arrow[r, "\varphi_{12}"] \arrow[d, "\varphi_{13}"'] & X_2 \arrow[dl, "\varphi_{23}"] \\
& & X_3 &
\end{tikzcd}$$

Morphisms $\mathcal{M}or(\{X, \psi_i\}, \{X', \psi'_i\})$, are morphisms $\Psi \in \mathcal{M}or(X, X')$, making the diagram formed by $(X, X', \psi_i, \psi'_i, \varphi_{ij})$ commutative. The terminal object in this category is called **limit**, or **inverse limit** of the diagram S.

DEFINITION: Colimit, or **direct limit** is obtained from the previous definition by inverting all arrows and replacing "terminal" by "initial".

Products and coproducts

EXERCISE: Prove that **the limit of $S = \{X_i, \varphi_{ij}\}$ is an object of $\mathcal{C}$ representing the contravariant functor $\mathcal{C}^\circ \longrightarrow \mathcal{S}ets$ which takes $X \in \mathcal{O}b(\mathcal{C})$ to the set of all morphisms $\psi_i : X \longrightarrow X_i$ making the above diagram commutative.**

EXAMPLE: Let S be a diagram with two vertices X_1 and X_2 and no arrows. The inverse limit of S is called **the product** of X_1 and X_2, and the direct limit **the coproduct**.

EXERCISE: Prove that **the product $X_1 \times X_2$ is an object which represents the functor $A \longrightarrow \mathcal{M}or(A, X_1) \times \mathcal{M}or(A, X_2)$ and coproduct is an object which represents the functor $A \longrightarrow \mathcal{M}or(X_1, A) \times \mathcal{M}or(X_2, A)$.**

EXAMPLE: Products in the category of sets, vector spaces and topological spaces are the usual products of sets, vector spaces and topological spaces **(check this)**.

EXAMPLE: Coproduct in the category of groups is called **free product**, or **amalgamated product**. Coproduct of the group $\mathbb{Z}$ with itself is called **the free group**. Coproduct in the category of vector spaces is also the usual product of vector spaces. Coproduct in the category of sets is disjoint union.

Preimage and diagonal (reminder)

Claim 2: Let $f: X \longrightarrow Y$ be a morphism of algebraic varieties, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ the corresponding ring homomorphism, $Z \subset Y$ a subvariety, and I_Z its ideal. Denote by R_1 the quotient of a ring $R := \mathcal{O}_X \otimes_{\mathcal{O}Y} (\mathcal{O}_Y/I_Z) = \mathcal{O}_X/f^*(I_Z)$ by its nilradical. **Then** $\operatorname{Spec}(R_1) = f^{-1}(Z)$**.**

Proof: Clearly, the set of common zeros of the ideal $J := f^*(I_Z)$ contains $f^{-1}(Z)$. On the other hand, for any point $x \in X$ such that $f(x) \notin Z$ there exist a function $g \in J$ such that $g(x) \neq 0$. Therefore, $f^{-1}(Z) = V_J$, and strong Nullstellensatz implies that $\mathcal{O}_{f^{-1}(Z)} = R_1$. ■

Claim 3: Let M be an algebraic variety, $\Delta \subset M \times M$ the diagonal, and $I \subset \mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M$ the ideal generated by $r \otimes 1 - 1 \otimes r$ for all $r \in \mathcal{O}_M$. **Then** $\mathcal{O}_\Delta$ **is** $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M/I$**.**

Proof. Step1: By definition of the tensor product, $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M/I = \mathcal{O}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M = \mathcal{O}_M$, hence it is reduced. If we prove that $\Delta = V_I$, the statement of the claim would follow from strong Nullstellensatz.

Step 2: Clearly, $\Delta \subset V_I$. To prove the converse, let $(m, m') \in M \times M$ be a point not on diagonal, and $f \in \mathcal{O}_M$ a function which satisfies $f(m) = 0, f(m') \neq 0$. Then $f \otimes 1 - 1 \otimes f$ is non-zero on (m, m'). ■

Fibered product (reminder)

DEFINITION: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be maps of sets. **Fibered product** $X \times_M Y$ is the set of all pairs $(x, y) \in X \times Y$ such that $\pi_X(x) = \pi_Y(y)$.

CLAIM: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be morphism of algebraic varieties, $R := \mathcal{O}_X \otimes_{\mathcal{O}_M} \mathcal{O}_Y$, and R_1 the quotient of R by its nilradical. **Then** $\operatorname{Spec}(R_1) = X \times_M Y$.

Proof: Let I be the ideal of diagonal in $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M$. Since I is generated by $r \otimes 1 - 1 \otimes r$ (Claim 3), $R = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_Y / (\pi_X \times \pi_Y)^*(I)$. Applying Claim 2, we obtain that $\operatorname{Spec}(R_1) = (\pi_X \times \pi_Y)^{-1}(\Delta)$. ■

Products and coproducts (2)

EXERCISE: Prove that **the product of algebraic varieties is their product in this category.**

EXERCISE: Prove that **coproduct of rings over $\mathbb{C}$ in the category of rings is their tensor product.**

EXERCISE: Prove that **coproduct of reduced rings over $\mathbb{C}$ in the category of reduced rings is the quotient of their tensor product by the nilradical.**

Since the category of algebraic varieties is equivalent to the category of finitely generated reduced rings, this gives another proof of the theorem.

THEOREM: Let A, B be finitely generated reduced rings over $\mathbb{C}$. **Then** $\operatorname{Spec}(A \otimes_{\mathbb{C}} B/I) = \operatorname{Spec}(A) \times \operatorname{Spec}(B)$, where I is nilradical.

Fibered product in categories

DEFINITION: Consider the following diagram:

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd}$$

Its limit is called **fibered product** of A and B over C. Colimit of the diagram

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

is called **coproduct** of A and B over C.

EXERCISE: Prove that the **fibered product of algebraic varieties is the same as their product in the category of algebraic varieties.**

EXERCISE: Prove that the **coproduct of rings A and B over C is $A \otimes_C B$.** Prove that the **coproduct of reduced rings A and B over C in the category of reduced rings is $A \otimes_C B/I$, where I is nilradical.**

Using strong Nullstellensatz again, we obtain
CLAIM: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be morphisms of affine varieties, $R := \mathcal{O}_X \otimes_{\mathcal{O}_M} \mathcal{O}_Y$, and R_1 the quotient of R by its nilradical. **Then** $\operatorname{Spec}(R_1) = X \times_M Y$**.**