Commutative Algebra

lecture 12: Normal varieties

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Zariski topology

DEFINITION: Zariski topology on an algebraic variety is a topology, where closed sets are algebraic subsets. **Zariski closure** of $Z \subset M$ is an intersection of all Zariski closed subsets containing Z.

DEFINITION: Cofinite topology is the topology on a set S such that the only closed subsets are S and finite sets.

EXERCISE: Prove that **Zariski topology on $\mathbb{C}$ coincides with the cofinite topology.**

CAUTION: Zariski topology is non-Hausdorff.

Zariski topology (2)

REMARK: We defined the Zariski topology on the set of points of A, that is, on the set of maximal ideals of $\mathcal{O}_A$ (this is how Zariski defined it). **Following Grothendieck, one defines the Zariski topology on the set $\operatorname{Spec}_{pr}(\mathcal{O}_A)$ of all prime ideals in $\mathcal{O}_A$: closed subsets Z_I in this topology correspond to prime ideals containing a given ideal $I \subset \mathcal{O}_A$.**

Oscar Zariski
(1899 – 1986)

Dominant morphisms

DEFINITION: **Dominant morphism** is a morphism $f: X \longrightarrow Y$, such that Y is a Zariski closure of $f(X)$.

PROPOSITION: Let $f: X \longrightarrow Y$ be a morphism of affine varieties. **The morphism f is dominant if and only if the homomorphism $\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$ is injective**.

Proof. Step1: If f^* is not injective, $f(X)$ lies in the set of common zeros of the ideal $\ker f^*$. Indeed, points of X are the same as maximal ideals and the same as homomorphisms $\mathcal{O}_X \longrightarrow \mathbb{C}$, and the points of $f(X)$ are homomorphisms $\mathcal{O}_Y \longrightarrow \mathbb{C}$ obtained as a composition $\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X \longrightarrow \mathbb{C}$.

Step 2: If $f(X)$ is contained in the set of common zeros of the ideal $J \subset \mathcal{O}_Y$, all functions $\alpha \in J$ vanish on $f(X)$. **This implies that $f^*(\alpha) = 0$.** ■

Field of fractions

DEFINITION: Let $S \subset R$ be a subset of R, closed under multiplication and not containing 0. **Localization** of R in S is a ring, formally generated by symbols a/F, where $a \in R$, $F \in S$, and relations $a/F \cdot b/G = ab/FG$, $a/F + b/G = \frac{aG+bF}{FG}$ and $aF^k/F^{k+n} = a/F^n$.

DEFINITION: Let R be a ring without zero divisors, and S the set of all non-zero elements in R. **Field of fractions** of R is a localization of R in S.

CLAIM: Let $f: X \longrightarrow Y$ be a dominant morphism, where X is irreducible. **Then Y is also irreducible**. Moreover, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ **can be extended to a homomorphism of the fields of fractions $k(Y) \longrightarrow k(X)$.**

Proof. Step1: Since $\mathcal{O}_Y$ is embedded in $\mathcal{O}_X$, and the later has no zero divisors, **$\mathcal{O}_Y$ has no zero divisors, hence Y is irreducible**.

Step 2: Embedding of rings without zero divisors can be extended to the fields of fractions: $f^*(a/F) = f^*(a)/f^*(F)$. ■

Principal divisors

DEFINITION: Let X be an affine variety, and $f \in \mathcal{O}_X$ a regular function which does not vanish on any of irreducible components of X. The zero set of f is called **a principal divisor** on X. Its irreducible components are called **divisors** on X.

REMARK: Let $X \subset \mathbb{C}^n$ be an affine variety, given by ideal $I \subset \mathbb{C}[x_1, ..., x_n]$, and $D \subset X$ be a divisor. **Then $X \backslash D$ is an affine variety, given by an ideal $I + \langle ft - 1 \rangle \subset \mathbb{C}[x_1, ..., x_n, t]$.**

DEFINITION: A dominant morphism of irreducible varieties is called **birational** if the corresponding homomorphism of the fields of fractions is an isomorphism.

EXAMPLE: **The natural map $X \backslash D \hookrightarrow X$ is birational.**

Birational morphisms

PROPOSITION: Let $f: X \longrightarrow Y$ be a birational morphism. **Then there exists a divisor $Z \subset Y$ such that $f: (X \backslash f^{-1}(Z)) \longrightarrow Y \backslash Z$ is an isomorphism.**

Proof. Step1: Since $\mathcal{O}_X$ is finitely generated, **there exists $F \in \mathcal{O}_Y$ such that for all $a \in \mathcal{O}_X$ there exists $b \in \mathcal{O}_Y$ such that $a = f\left(\frac{b}{F^k}\right)$.** Indeed, for each generator x_i of $\mathcal{O}_X$ there exists $y_i, F_i \in \mathcal{O}_Y$ such that $x_i = f\left(\frac{y_i}{F_i}\right)$. Choosing $F = \prod F_i$, we obtain that $x_i = f\left(\frac{y_i'}{F}\right)$, where $y_i' = y_i \prod_{j \neq i} F_j$. Then for each homogeneous polynomial $P(x_i, ..., x_j)$ of degree d we have $P(x_1, ..., x_n) = f\left(\frac{P(y_1', ... y_n')}{F^d}\right)$.

Step 2: Let Z be the set of all $p \in Y$ such that $F(p) = 0$. **Then F is invertible in $Y \backslash Z$, hence $f^{-1}: Y \backslash Z \longrightarrow X$ is a polynomial map.** Therefore, $f: (X \backslash f^{-1}(Z)) \longrightarrow Y \backslash Z$ is invertible. ∎

Integral dependence

DEFINITION: Let $A \subset B$ be rings. An element $b \in B$ is called **integral over A** if the subring $A[b] = A \cdot \langle 1, b, b^2, b^3, ... \rangle$, generated by b and A, is finitely generated as A-module.

DEFINITION: Monic polynomial is a polynomial with leading coefficient 1.

REMARK: When A is Noetherian, the following statement is an equivalence of two characterizations of Noetherian A-modules. However, it is true without the Noetherian assumtion.

CLAIM: An element $x \in B$ **is integral over $A \subset B$ if and only if the chain of submodules**

$$A \subset A \cdot \langle 1, x \rangle \subset A \cdot \langle 1, x, x^2 \rangle \subset A \cdot \langle 1, x, x^2, x^3 \rangle \subset ...$$

terminates.

Proof: If the chain terminates, then $A[x]$ is clearly finitely generated. Conversely, if $A[x]$ is finitely generated, any degree of x can be expressed through a finite number of generators, which can be expressed as polynomials on x. ■

COROLLARY: An element $x \in B$ is integral over $A \subset B \Leftrightarrow x$ is a root of a monic polynomial with coefficients in A. ■

Sum and product of integral elements is integral

EXERCISE: Let $x, y \in B \supset A$, with x integral over A and y integral over $A[x]$. **Prove that y is integral over A.**

CLAIM: Let $A \subset B$ be Noetherian rings. Then **sum and product of elements which are integral over A is also integral.**

Proof: Let $x, y \in B$ be integral over A. Since y is integral over $A[x]$, which is finitely generated as A-module, the ring $A[x, y]$ is finitely generated as an A-module. Since A is Noetherian, a submodule of finitely-generated A-module is finitely-generated, hence $x + y$ and $xy \in A[x, y]$ are also integral. ■

Integral closure

DEFINITION: Let $A \subset B$ be rings. The set of all elements in B which are integral over A is called **the integral closure of A in B**.

DEFINITION: Let A be the ring without zero divisors, and $k(A)$ its field of fractions. The set of all elements $a \in k(A)$ which are integral over A is called **the integral closure of A**. A ring A is called **integrally closed** if A coincides with its interal closure in $k(A)$.

REMARK: As shown above, **the integral closure is a ring.**

DEFINITION: An affine variety X is called **normal** if all its irreducible components X_i are disconnected, and the ring of functions $\mathcal{O}_{X_i}$ for each of these irreducible components is integrally closed.

REMARK: Equivalently, **X is normal if any finite, birational morphism $Y \longrightarrow X$ is an isomorphism.**

Factorial rings

DEFINITION: An element p of a ring R is called **indecomposable** if for any decomposition $p = p_1 p_2$, either p_1 or p_2 is invertible.

DEFINITION: A ring R without zero divisors is called **factorial** if any element $r \in R$ can be represented as a product of indecomposable elements, $r = \prod_i p_i^{\alpha_i}$, and this decomposition is unique up to invertible factors and permutation of p_i.

PROPOSITION: Let A be a factorial ring. Then it is integrally closed.

Proof. Step1: Let $u, v \in A$, and $u/v \in k(A)$ a root of a monic polynomial $P(t) \in A[t]$ of degree n. **Then u^n is divisible by v in A.**

Step 2: Let $u/v \in k(A)$ be a root of a monic polynomial $P(t) \in A[t]$. Assume that u, v are comprime. **Since u^n is divisible by v, and they are coprime, v is invertible by factoriality of A.** Then $u/v \in A$. ■