Commutative Algebra

lecture 14: Nakayama lemma

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Nakayama's lemma

Tadashi Nakayama (1912-1964)

QUESTION: Let $\mathfrak{a} \subset A$ be a non-trivial ideal in a Noetherian ring. **How can we prove that $\bigcap_i \mathfrak{a}^i = 0$?**

ANSWER: **Nakayama's lemma!**

REMARK: $\bigcap_i \mathfrak{a}^i = 0$ **does not hold** in the ring of smooth functions, which is non-Noetherian.

DEFINITION: An A-module M is called **torsion-free** if for any non-zero $a \in A$, and any non-zero $m \in M$, one has $am \neq 0$.

Nakayama's lemma: Let A be a Noetherian ring, and M a finitely-generated torsion-free A-module. **Then for any non-trivial ideal $\mathfrak{a} \subset A$, $\mathfrak{a}M = M$ implies $M = 0$.**

Nakayama's lemma (2)

Nakayama's lemma: Let A be a Noetherian ring, and M a finitely-generated torsion-free A-module. Assume that the annihilator $\operatorname{Ann}_M := \{a \in A \mid aM = 0\}$ vanishes. **Then for any non-trivial ideal $\mathfrak{a} \subset A$, $\mathfrak{a}M = M$ implies $M = 0$.**

Proof. Step1: For any finitely-generated A-module M over a Noetherian ring, $\operatorname{End}_A(M)$ is finitely-generated as an A-module **(prove it as an exercise)**.

Step 2: For any $\Phi \in \operatorname{End}_A(M)$, consider the subalgebra $A[\Phi] \subset \operatorname{End}_A(M)$, generated by Φ. Since $\operatorname{End}_A(M)$ is finitely generated, $A[\Phi]$ is Noetherian. Therefore, Φ^n is expressed as a sum $\sum_{i=0}^{n-1} a_i\Phi^i$ for n sufficiently big. We obtain that **Φ is a root of monic polynomial with coefficients in A.**

Step 3: Let $\Phi \in \operatorname{End}_A(M)$, $e_1, ..., e_n$ generators of M, and (a_{ij}) the matrix of Φ written in this basis **(it is non-unique)**. Define **characteristic polynomial** of Φ as $\operatorname{Chpoly}_\Phi(t) := \det(t\operatorname{Id} - A)$, where $A = (a_{ij})$.

Nakayama's lemma (2)

Step 3: Let $\Phi \in \operatorname{End}_A(M)$, $e_1, ..., e_n$ generators of M, and (a_{ij}) the matrix of Φ written in this basis **(it is non-unique)**. Define **characteristic polynomial** of Φ as $\operatorname{Chpoly}_\Phi(t) := \det(t \operatorname{Id} - A)$, where $A = (a_{ij})$.

Step 4: Cayley-Hamilton theorem gives $\operatorname{Chpoly}_\Phi(\Phi) = 0$ for any endomorphism of a finite-dimensional space over a field k. Then the same is true for a free module over any subring $R \subset k$, in particular, for a polynomial ring. However, any ring is a quotient of (possibly infinitely generated) polynomial ring, hence $\operatorname{Chpoly}_\Phi(\Phi) = 0$ is true for any endomorphism of a free, finitely-generated A-module. Since any finitely-generated module is a quotient of a free module, **we have $\operatorname{Chpoly}_\Phi(\Phi) = 0$ for any $\Phi \in \operatorname{End}_A(M)$.**

Step 5: Let $\operatorname{Chpoly}_{\operatorname{Id}_M}(t) = t^n + \sum_{i=0}^{n-1} a_i t^i$ be the characteristic polynomial for the identity map $\operatorname{Id}_M \in \operatorname{End}_A(M)$. This polynomial depends on the choice of generators of M. **Cayley-Hamilton give $\operatorname{Chpoly}_{Id}(\operatorname{Id}) = 0$, hence $(\sum a_i + 1) = 0$ in the ring $\operatorname{End}_A(M)$.** The ring $\operatorname{End}_A(M)$ contains A, because $\operatorname{Ann}_A M = 0$. **Then $1 = -\sum a_i$ in A. This is the only place we use the assumption $\operatorname{Ann}_A M = 0$.**

Step 6: If $\mathfrak{a}M = M$, the identity map can be represented by a matrix (a_{ij}) with $a_{ij} \in \mathfrak{a}$. Step 5 gives $\sum a_i + 1 = 0$, which is impossible. ■

Krull theorem

Wolfgang Krull (1899-1971), Göttingen, 1920

THEOREM: (Krull theorem)
Let $\mathfrak{a} \subset A$ be an ideal in a Noetherian ring without zero divisors. Then $\bigcap \mathfrak{a}^n = 0$.

Proof: Let $M := \bigcap \mathfrak{a}^n$. This is a torsion-free module satisfying $\mathfrak{a}M = M$. Nakayama's lemma implies $M = 0$. ■

REMARK: A version of Nakayama's lemma is valid for all A-modules, regardless of torsion.

THEOREM: (Nakayama's lemma)
Let A be a Noetherian ring, and M a finitely-generated A-module. **Then for any non-trivial ideal $\mathfrak{a} \subset A$, $\mathfrak{a}M = M$ implies that $(1 + a)M = 0$,** for some $a \in \mathfrak{a}$.

Proof: Let $\operatorname{Ann}_M := \{a \in A \mid aM = 0\}$. Replace A by A/Ann_M and apply Step 5 of Nakayama Lemma. ■

Local rings

DEFINITION: A ring A is called **local** if it has only one maximal ideal.

DEFINITION: Let $\mathfrak{p} \subset A$ be a prime ideal, and $S \subset A$ its complement. **Localization of A in $\mathfrak{p}$** is $A[S^{-1}]$.

CLAIM: Localization of A in $\mathfrak{p}$ is local.

Proof: Any $x \in A \backslash \mathfrak{p}$ is invertible, hence $\mathfrak{p}$ **is maximal ideal containing all ideals in A.** ■

CLAIM: Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, and $\Phi \in \operatorname{Hom}_A(M_1, M_2)$ a homomorphism of finitely-generated A-modules. Suppose that Φ induces a surjective map $\operatorname{Hom}_{A/\mathfrak{m}}(M_1/\mathfrak{m}M_1, M_2/\mathfrak{m}M_2)$. **Then Φ is surjective.**

Proof: Let $M_3 := \operatorname{coker} \Phi$. For any $x \in M_2$, one has $x \in \operatorname{im} \Phi \mod \mathfrak{m}$. Therefore, $\mathfrak{m}M_3 = M_3$. Then Nakayama's lemma implies that $(1+a)M_3 = 0$, for some $a \in \mathfrak{m}$. Since $1+a$ is invertible, this implies that $M_3 = 0$. ■

Finite morphisms (reminder)

REMARK: Let M be a finitely generated R-module, and $R \longrightarrow R'$ a ring homomorphism. **Then $M \otimes_R R'$ is a finitely generated R'-module**. Indeed, **if M is generated by $x_1, ..., x_n$, then $M \otimes_R R'$ is generated by $x_1 \otimes 1, ..., x_n \otimes 1$.**

DEFINITION: A morphism $X \longrightarrow Y$ of affine varieties is called **finite** if the ring $\mathcal{O}_X$ is a finitely generated module over $\mathcal{O}_Y$. In this case, $\mathcal{O}_X$ is called **an integral extension** of $\mathcal{O}_Y$.

THEOREM: Let $X \xrightarrow{f} Y$ be a finite morphism. Then for any point $y \in Y$, **the preimage $f^{-1}(y)$ is finite.**

Proof. Step1: Since $\mathcal{O}_X$ is finite generated as an $\mathcal{O}_Y$-module, the ring $R := \mathcal{O}_X \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathfrak{m}_y)$ is finitely generated as an $\mathcal{O}_Y/\mathfrak{m}_y$-module. Since $\mathcal{O}_Y/\mathfrak{m}_y = \mathbb{C}$, we obtain that R is an Artinian algebra over $\mathbb{C}$.

Step 2: Let $N \subset R$ be a nilradical. As shown in Lecture 11, Spec(R/N) **is a finite set.**

Step 3: As shown in Lecture 10, Spec$(R/N) = f^{-1}(y)$. ■

Dominant, finite morphisms are surjective

THEOREM: Let $f : X \longrightarrow Y$ be a finite, dominant morphism of affine varieties. **Then f is surjective.**

Proof. Step1: Restricting to irreducible components, we can always assume that Y, and hence X is irreducible. Let $A = \mathcal{O}_Y$, $B = \mathcal{O}_X$. We can consider A as a subring of B, which has no zero divisors, and assume that B is finitely generated as A-module.

Step 2: Let $\mathfrak{m}_y \subset A$ be a maximal ideal corresponding to $y \in Y$. **Nakayama's lemma implies that $\mathfrak{m}_y B \neq B$.**

Step 3: $f^{-1}(y) = \operatorname{Spec}(B \otimes_A A/\mathfrak{m}_y) = \operatorname{Spec}(B/\mathfrak{m}_y B)$. **Since this is non-zero ring, the set $f^{-1}(y)$ is non-empty.** ■