Commutative Algebra

lecture 15: Finite quotients and branched covers

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Finite morphisms (reminder)

REMARK: Let M be a finitely generated R-module, and $R \longrightarrow R'$ a ring homomorphism. **Then $M \otimes_R R'$ is a finitely generated R'-module.** Indeed, **if M is generated by $x_1, ..., x_n$, then $M \otimes_R R'$ is generated by $x_1, ..., x_n$.**

DEFINITION: A morphism $X \longrightarrow Y$ of affine varieties is called **finite** if the ring $\mathcal{O}_X$ is a finitely generated module over $\mathcal{O}_Y$. In this case, $\mathcal{O}_X$ is called **an integral extension** of $\mathcal{O}_Y$.

THEOREM: Let $X \xrightarrow{f} Y$ be a finite morphism. Then for any point $y \in Y$, **the preimage $f^{-1}(y)$ is finite.**

Proof. Step1: Since $\mathcal{O}_X$ is finite generated as an $\mathcal{O}_Y$-module, the ring $R := \mathcal{O}_X \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathfrak{m}_y)$ is finitely generated as an $\mathcal{O}_Y/\mathfrak{m}_y$-module. Since $\mathcal{O}_Y/\mathfrak{m}_y = \mathbb{C}$, we obtain that R is an Artinian algebra over $\mathbb{C}$.

Step 2: Let $N \subset R$ be a nilradical. As shown in Lecture 9, Spec(R/N) **is a finite set.**

Step 3: As shown in Lecture 8, Spec$(R/N) = f^{-1}(y)$. ■

Dominant, finite morphisms are surjective (reminder)

THEOREM: Let $f: X \longrightarrow Y$ be a finite, dominant morphism of affine varieties. **Then f is surjective.**

Proof. Step1: Restricting to irreducible components, we can always assume that Y, and hence X is irreducible. Let $A = \mathcal{O}_Y$, $B = \mathcal{O}_X$. We can consider A as a subring of B, which has no zero divisors, and assume that B is finitely generated as A-module.

Step 2: Let $\mathfrak{m}_y \subset A$ be a maximal ideal corresponding to $y \in Y$. **Nakayama's lemma implies that $\mathfrak{m}_y B \neq B$.**

Step 3: $f^{-1}(y) = \operatorname{Spec}(B \otimes_A A/\mathfrak{m}_y) = \operatorname{Spec}(B/\mathfrak{m}_y B)$. **Since this is non-zero ring, the set $f^{-1}(y)$ is non-empty.** ■

Finite quotients

CLAIM: Let R be a Noetherian ring without zero divisors, G a finite group acting by automorphisms on R, and R^G the ring of G-invariants. **Then $\varphi:$ Spec $R \longrightarrow$ Spec R^G is a finite, dominant morphism.**

Proof. Step1: For any $g \in G$, consider the corresponding polynomial map $P_g : R \longrightarrow R$, and let $r \in R$. The polynomial $P(t) := \prod_{g\in G}(t - g(r))$ has G-invariant coefficients for any $r \in R$, hence $P(t) \in R^G[t]$

Step 2: The morphism φ is finite because each $r \in R$ satisfies the equation $P(r) = 0$, where $P(t) = \prod_{g\in G}(t - g(r))$. It is dominant, because $R^G \subset R$. ■

DEFINITION: Let G be a finite group acting on an affine variety X by automorphisms. **The quotient space** X/G is Spec$(\mathcal{O}_X^G)$.

EXAMPLE: $\mathbb{C}^2/\{\pm 1\} = \mathbb{C}[x^2, y^2, xy] = \mathbb{C}[t_1, t_2, t_3]/(t_1t_2 = t_3^2)$. Indeed, $\mathbb{C}^2/\{\pm 1\} =$ Spec A, where $A = \mathbb{C}[x, y]^{\{\pm 1\}}$: **$A$ is the ring of even polynomials**.

EXAMPLE: Let $G = \mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}$ by multiplication by a primitive root $\sqrt[n]{1}$. Then $\mathbb{C}/G =$ Spec$(\mathbb{C}[t]^G) =$ Spec$(\mathbb{C}[t^n])$, hence **the quotient space $\mathbb{C}/G$ is isomorphic to $\mathbb{C}$**.

Finite quotients (2)

THEOREM: Consider the natural morphism $\operatorname{Spec} R \xrightarrow{\varphi} \operatorname{Spec} R^G$. Then $\varphi(x) = \varphi(y)$ **if and only if** $x \in G \cdot y$, that is, the **set of points in** $\operatorname{Spec} R^G$ **is identified with the space of** G**-orbits**.

Proof. Step1: If two maximal ideals of R are G-conjugated, their intersections of $R^G \subset R$ are equal. This gives $\varphi(gx) = \varphi(x)$: **each** G**-orbit is mapped to one point**. It remains to show that **the preimage of any point is exactly one** G**-orbit.**

Step 2: For any ideal $\mathfrak{m} \subset R^G$, one has $(\mathfrak{m}R)^G = \mathfrak{m}$ (lecture 6). Then $A^G = R^G/\mathfrak{m}$, where $A := R \otimes_{R^G} (R^G/\mathfrak{m}) = R/\mathfrak{m}R$.

Step 3: Let $\mathfrak{m}$ be the maximal ideal of $y \in \operatorname{Spec} R^G$, and N the nilradical of $A := R/\mathfrak{m}R$. Since $\varphi^{-1}(y) = \operatorname{Spec}(A/N)$, **points of** $\varphi^{-1}(y)$ **are maximal ideals of the ring** A/N.

Step 4: A semisimple Artinian $\mathbb{C}$-algebra A/N is a direct sum of finite extensions of $\mathbb{C}$, which are all isomorphic to $\mathbb{C}$, giving $A/N = \bigoplus \mathbb{C}$. Since $A^G = \mathbb{C}$ (Step 2), **the group** G **acts on the summands of** $A/N = \bigoplus \mathbb{C}$ **transitively**. Therefore, **all points of** $\varphi^{-1}(y)$ **belong to the same** G**-orbit**. ■

Transcendental extensions

CLAIM: Let $[k:\mathbb{C}]$ be an extension of $\mathbb{C}$, and $[K:k]$ an extension of k generated over k by $z\in K$. **Then either z is transcendental, and K is isomorphic to the field of rational functions $k(z)$, or z is algebraic, and $[K:k]$ is a finite extension.**

Proof: Indeed, either z is a root of polynomial, and then $[K:k]$ is finite, or K contains the polynomial ring $k[z]$, and then K contains $k(z)$. ■

LEMMA: Let $[K:k]$ be a finite extension, and $[K(t):k(t)]$ the correspoinding extension of rational functions. **Then $[K(t):k(t)]$ is also finite**.

Proof: Primitive element theorem gives $K=k[\alpha]$, where α is an algebraic element, which is a root of a polynomial $P(z)$. Using the isomorphism $A/J\otimes_A B=B/JB$ (Lecture 7), we obtain

$$K(t)=K\otimes_k k(t)=\frac{k[z]}{(P(z))}\otimes_k k(t)=k(t)[z]/(P(z)).$$

■

EXERCISE: Find a proof which does not use existence of a primitive element.

Transcendence basis

DEFINITION: Let $k(t_1, ..., t_n)$ be the field of rational functions of several variables, that is, the fraction field for the polynomial ring $k[t_1, ..., t_n]$. Then the extension $[k(t_1, ..., t_n) : k]$ is called **a purely transcendental extension of k**, and $t_1, ..., t_n$ are called **algebraically independent**.

REMARK: Clearly, $t_1, ..., t_n$ **are algebraically independent if and only if there are no alrebraic relations of form $P(t_1, ..., t_n) = 0$,** where P is a polynomial of n variables.

DEFINITION: **Transcendence basis** of an extension $[K : k]$ is a collection $z_1, ..., z_n \in K$ generating a purely trascendental extension $K' := k(z_1, ..., z_n)$ such that $[K : K']$ is an algebraic extension.

Transcendence basis in regular functions

Theorem 1: Let $k(X)$ be the field of rational functions on an irreducible affine variety X, with $\mathcal{O}_X$ generated by $t_1, ..., t_n$. Let $S \subset \{t_1, ..., t_n\}$ be a maximal algebraically independent subset. **Then the extension $[k(X) : k(t_1, ..., t_k)]$ is finite**.

Proof: Since $\mathcal{O}_K$ is finitely generated, we can use induction by the number of generators $t_1, ..., t_n$ of $\mathcal{O}_X$. Let $A \subset \mathcal{O}_X$ be a subring generated by $t_1, ..., t_{n-1}$, and $t_1, ..., t_k$ a transcendence basis on $k(A)$.

If t_n is algebraic over $k(A)$, then $[k(X) : k(A)]$ is finite; since $[k(A) : k(t_1, ..., t_k)]$ is finite, this implies that $[k(X) : k(t_1, ..., t_k)]$ is finite.

If t_n is transcendental over $k(A)$, we obtain $k(X) = k(A)(t_n)$, and $[k(X)] = k(A)(t_n)$ is finite over $k(t_1, ..., t_k, t_n)$ by the lemma above. ■

Transcendence basis and dominant morphisms

PROPOSITION: Let $X \subset \mathbb{C}^n$ be an irreducible affine manifold, $t_1, ..., t_n$ coordinates on $\mathbb{C}^n$, and $\Pi_k : X \longrightarrow \mathbb{C}^k$ the projection to the first k coordinates. **Then the following are equivalent.**

(i) Π_k is dominant and the extension $[k(X) : k(t_1, ..., t_k)]$ is finite.

(ii) $t_1, ..., t_k$ is transcendence basis in $k(X)$.

Proof: Theorem 1 implies that $[k(X) : k(t_1, ..., t_k)]$ is finite whenever $t_1, ..., t_k$ is the transcendence basis. Therefore, (ii) $\Rightarrow$ (i). Converse is clear, because $k(X)$ is algebraic over $k(t_1, ..., t_k) \subset k(X)$, hence $t_1, ..., t_k$ is a maximal algebraically independent subset. ■

REMARK: We want to find a projection $\Pi_k : X \longrightarrow \mathbb{C}^k$ which is finite, for any given irreducible affine variety X.

When the coordinate projection is finite

REMARK: Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, z_i coordinates on $\mathbb{C}^n$, and $z_1, ..., z_k$ transcendence basis on $k(X)$. **The projection map Π_{n-1} is finite if and only if $P(z_n) = 0$ in $\mathcal{O}_X$, for some monic polynomial $P(t) \in \mathcal{O}_X[t]$ with coefficients which are polynomial in $z_1, ..., z_{n-1}$.** Indeed, this is precisely what is needed for $\mathcal{O}_X$ to be a finitely generated module over its subalgebra $A = \mathcal{O}_{P_{n-1}(X)}$ generated by $z_1, ..., z_{n-1}$ considered as regular functions on X. Notice that **a non-zero polynomial with $P(t) \in A[t]$ with $P(z_n) = 0$ always extists, unless $n = k$ and $X = \mathbb{C}^n$,** but it is not necessarily monic.

CLAIM: In these assumptions, **there exists a linear coordinate change $z_i' := z_i + \lambda_i z_n$, such that z_n is finite over $z_1', ..., z_k'$,** that is, z_n is a root of monic polynomial with coefficients in $\mathbb{C}[z_1', ..., z_k']$.

When the coordinate projection is finite (2)

CLAIM: In these assumptions, **there exists a linear coordinate change $z_i' := z_i + \lambda_i z_n$, such that z_n is finite over $z_1', ..., z_k'$.**

Proof. Step1: Let $P(z_1, ..., z_k, z_n) = 0$ be a non-zero polynomial relation in $\mathcal{O}_X$, of positive degree in z_n. Such a relation exists because $z_1, ..., z_k$ is a transcendence basis in $k(X)$, and z_n is algebraic over $z_1, ..., z_k \in \mathcal{O}_X$. Choose P which has minimal possible degree in $z_1, ..., z_k, z_n$. Let $F_P(z_1, ..., z_k, z_n)$ be a homogeneous component of top degree d in $P(z_1, ..., z_k, z_n)$.

Step 2: Consider a polynomial

$$Q(z_1', ..., z_k', z_n) := F(z_1 + \lambda_1 z_n, ..., z_k + \lambda_k z_n, z_n).$$

Then $Q(0, 0, ..., 0, 1) = F(\lambda_1, ..., \lambda_k, 1)$ is non-zero on X for general λ_i. Indeed, if $F(\lambda_1, ..., \lambda_k, 1)$ is identically 0 for all λ_i, the top degree homogeneous term in $P(z_1, ..., z_k, z_n)$ vanishes on X, and we can replace $P(z_1, ..., z_k, z_n)$ by a smaller degree polynomial.

Step 3: The polynomial $Q(z_1', ..., z_k', t)$ is homogeneous of degree d, which is the maximal degree of all terms in $P(z_1, ..., z_k, z_n)$. The degree d polynomial $P(z_1', ..., z_k', t)$ is monic in t, because its leading term t^d has non-zero scalar coefficient by Step 2, hence **z_d is finite over $z_1', ..., z_k'$.** ■

Noether's normalization lemma, first version

REMARK: Let $A \subset B \subset C$ be ring without zero divisors, C is finitely generated as B-module, and B as an A-module. Then **C is finitely generated as an A-module.**

REMARK: We have proven that the projection $\Pi_{n-1} : X \longrightarrow \mathbb{C}^{n-1}$ to coordinates $z'_1, ..., z'_k, z_{k+1}, ..., z_{n-1}$ is finite. Using this remark and induction by n, we obtain

COROLLARY: (Noether's normalization lemma, first version)
Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety. Then there exists a linear coordinate change such that **the projection of X to the first k coordinates gives a finite map $X \longrightarrow \mathbb{C}^k$.** ■

REMARK: This is expressed by saying that **the projection of X to the first k coordinates is a branched cover.**