Commutative Algebra

lecture 16: Noether's normalization lemma

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Noether's normalization lemma (first version)

PROPOSITION: Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, z_i coordinates on $\mathbb{C}^n$, and $z_1, ..., z_k$ transcendence basis on $k(X)$. Then, for all $\lambda_1, ..., \lambda_k$ outside of the zero-set of a certain non-zero homogeneous polynomial, **the function $z_n \in \mathcal{O}_X$ is a root of a monic polynomial in the variables $z'_1, ..., z'_k$, where $z'_i := z_i + \lambda_i z_n$.**

Proof: Lecture 14. ∎

Corollary 1: **(Noether's normalization lemma, first version)**
Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, z_i coordinates on $\mathbb{C}^n$, and $z_1, ..., z_k$ transcendence basis on $k(X)$. Then **there exists a linear coordinate change $z'_i := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$, such that the projection $\Pi_k : X \longrightarrow C^k$ to the first k arguments is a finite, dominant morphism.**

Proof: Previous proposition shows that the projection $P_n : X \longrightarrow \mathbb{C}^{n-1}$ is finite onto its image X_1 (after some linear adjustment). Using induction by n, we can assume that $P_k : X_1 \longrightarrow \mathbb{C}^k$ is also finite, hence the composition map is finite **(composition of finite morphisms is always finite, as we have seen).** ∎

Noether's normalization lemma for non-irreducible varieties

The following version works for non-irreducible varieties.

PROPOSITION: Let $X \subset \mathbb{C}^n$ be an affine subvariety, and X_i its irreducible components. Denote by k the maximal transcendence degree for $k(X_i)$. Then **there exists a linear coordinate change $z_i' := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$, such that the projection $\Pi_k: X \longrightarrow C^k$ to the first k arguments is a finite.**

Proof. Step 1: The natural projection map

$$\Psi: \mathcal{O}_X \longrightarrow \prod_{\mathfrak{m} \in \operatorname{Spec}(\mathcal{O}_X)} \mathcal{O}_X/\mathfrak{m}$$

is injective by Hilbert Nullstellensatz.

Step 2: The natural projection map $\Phi: \mathcal{O}_X \longrightarrow \bigoplus \mathcal{O}_{X_i}$ is injective, because Ψ factorizes through Φ. It is also finite, because $\mathcal{O}_{X_i}$ is finitely generated over $\mathcal{O}_X$. Clearly, $\coprod X_i = \operatorname{Spec}(\bigoplus \mathcal{O}_{X_i})$, where $\coprod$ denotes the disjoint union.

Step 3: Choose a coordinate projection $\Pi_k: \mathbb{C}^n \longrightarrow C^k$ which is finite on each X_i; such a projection exists by Corollary 1. The composition $\coprod X_i \longrightarrow X \xrightarrow{\Pi_k} \mathbb{C}^k$ is finite, hence $\bigoplus \mathcal{O}_{X_i}$ is a finitely generated $\mathcal{O}_{\mathbb{C}^k}$-module. Since $\mathcal{O}_{\mathbb{C}^k}$ is Noetherian, the submodule $\mathcal{O}_X \subset \bigoplus \mathcal{O}_{X_i}$ is also finitely generated. ■

Integral closure is finite

DEFINITION: Let $A \subset B$ be rings. The set of all elements in B which are integral over A is called **the integral closure of A in B**.

REMARK: The ring $\mathbb{C}[z_1, ..., z_n]$ **is factorial by Gauss lemma,** and therefore **integrally closed.**

THEOREM: Let A be an integrally closed Noetherian ring, $[K : k(A)]$ a finite extension of its field of fractions, and B the integral closure of A in K. **Then B is finitely generated as an A-module**.

Proof: Proven in Lecture 13. ■

EXAMPLE: Let $[K : \mathbb{C}(z_1, ..., z_n)]$ be a finite extension. **Then the integral closure of $C[z_1, ..., z_n]$ in X is finitely generated.**

Normalization

COROLLARY: Let X be an affine variety, and $\hat{A}$ the integral closure of its ring of regular functions. **Then the ring $\hat{A}$ is finitely generated.**

Proof. Step 1: The variety X admits a finite, dominant map to $\mathbb{C}^k$. Let A be the integral closure of $\mathbb{C}[z_1, ..., z_n]$ in $k(X)$; it is a finitely generated algebra by the previous theorem. Then A is an integrally closed ring containing $\mathcal{O}_X$ and with the same field of fractions.

Step 2: It remains to show that $A = \hat{A}$. Since $A \supset \mathcal{O}_X \supset \mathbb{C}[z_1, ..., z_n]$, we obtain that A is a finitely generated module over $\mathcal{O}_X$. Therefore, $A \subset \hat{A}$; since $A \supset \mathcal{O}_X$ us integrally closed, A contains the integral closure of $\mathcal{O}_X \subset k(X)$, which gives $A \supset \hat{A}$. ■

DEFINITION: Let X be an affine variety, and $\hat{A}$ the integral closure of its ring of regular functions. Then $\hat{X} :=$ Spec$(\hat{A})$ is called **normalization of X**.

Normalization (2)

DEFINITION: Let X be an affine variety, and $\hat{A}$ the integral closure of its ring of regular functions. Then $\hat{X} := \operatorname{Spec}(\hat{A})$ is called **normalization of X**.

REMARK: The normalization map is finite and birational; **X is normal if for any finite, birational $\varphi: X' \longrightarrow X$, the map φ is an isomorphism**. Indeed, in this case $\mathcal{O}_{X'} \supset \mathcal{O}_X$ is finite with the same field of fractions.

COROLLARY: Normalization of X is a finite, birational morphism $X' \longrightarrow X$ **such that for any other finite, birational $\varphi: X'' \longrightarrow X'$, the map φ is an isomorphism**. In particular, **any birational, finite map $X' \longrightarrow X$ with X' normal is a normalization.** ■

REMARK: In other words, normalization **is an initial object in the category $\mathcal{C}$, such that $\mathcal{O}b(\mathcal{C})$ is the set of the pairs** $(X' \xrightarrow{\mu'} X)$ where the morphism μ is finite and birational, and morphisms of $\mathcal{C}$ are maps $\varphi: X' \longrightarrow X''$ making the following diagram commutative

$$\begin{tikzcd} X' \arrow[rr, "\varphi"] \arrow[dr, "\mu'"'] & & X'' \arrow[dl, "\mu''"] \\ & X & \end{tikzcd}$$

Finite union of vector spaces over infinite fields

Proposition 1: Let $V = k^n$ be a vector space over a field k of characteristic 0, and $W_1, ..., W_n \subsetneq V$ proper subspaces. **Then $V \neq \bigcup W_i$.**

Proof. Step 1: Replacing W_i by a bigger subspace if necessarily, we can assume all W_i have codimension 1 and are defined by an equation $\lambda_i(v) = 0$. Then $X := \bigcup W_i \subset V$ is an affine subvariety which is given by an equation $\prod \lambda_i = 0$.

Step 2: Let $z_1, ..., z_n$ be coordinates in V, and $z_1, ..., z_k \in k(X)$ a transcendence basis (renumber z_i if necessarily so that algebraically independent coordinates go first). The equation $\prod \lambda_i = 0$ gives an algebraic relation between z_i, restricted to X. **Therefore $k < n$.**

Step 3: After an appropriate linear change, we find a linear projection $\Pi : W \longrightarrow W_1$, with $\dim W_1 = k$, such that $\Pi : X \longrightarrow W_1$ is finite (Noether normalization lemma).

Step 4: **The fibers of $\Pi : X \longrightarrow W_1$ are finite,** but the fibers of $\Pi : W \longrightarrow W_1$ are vector spaces, and they are infinite. ■

Primitive element theorem (reminder)

LEMMA: Let k be a field, and $A := \bigoplus_{i=1}^n k$. **Then A contains only finitely many different k-algebras.**

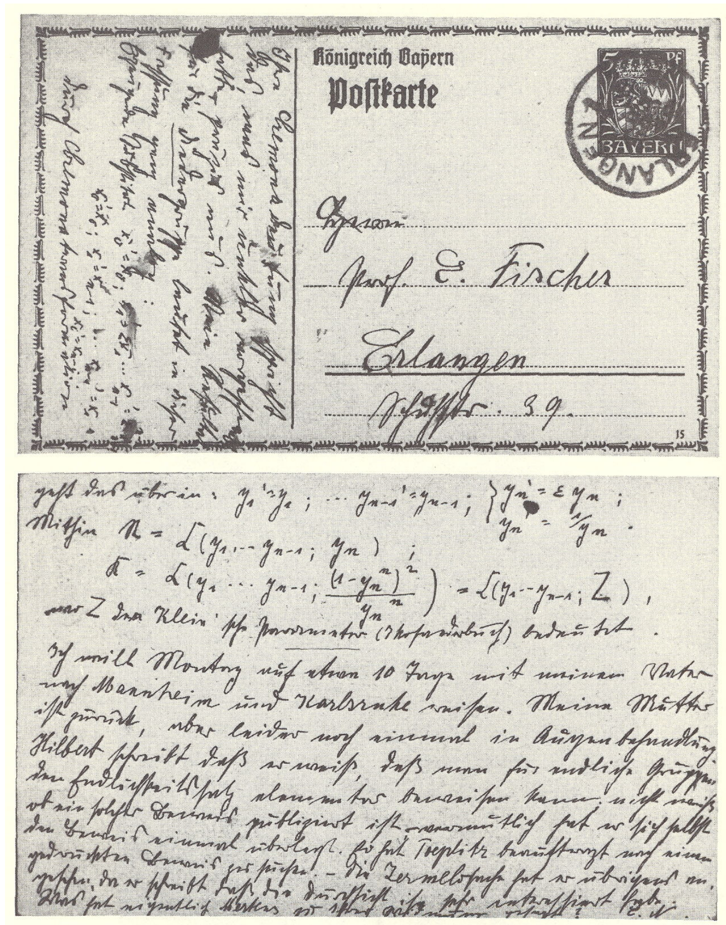
Proof: Let $e_1, ..., e_n$ be the units in the summands of A. Then any udempotent $a \in A$ is a sum of udempotents $a = \sum e_i a$, but $e_i a$ belongs to the i-th summand of A. Then $e_i a = 0$ or $e_i a = e_i$, because k contains only two udempotents. This implies that **any k-algebra $A_i \subset A$ is generated by an idempotent a, which is sum of some a_i.** ■

THEOREM: Let $[K : k]$ be a finite field extension in $\operatorname{char} = 0$. **Then there exists a primitive element $x \in K$,** that is, an element which generates K.

Proof. Step 1: Let $\bar{k}$ be the algebraic closure of k. **The number of intermediate fields $K \supset K' \supset k$ is finite.** Indeed, all such fields correspond to $\bar{k}$-subalgebras in $K \otimes_k \bar{k}$, and **there are finitely many k-subalgebras in $K \otimes_k \bar{k}$ because $K \otimes_k \bar{k} = \bigoplus_i \bar{k}$.**

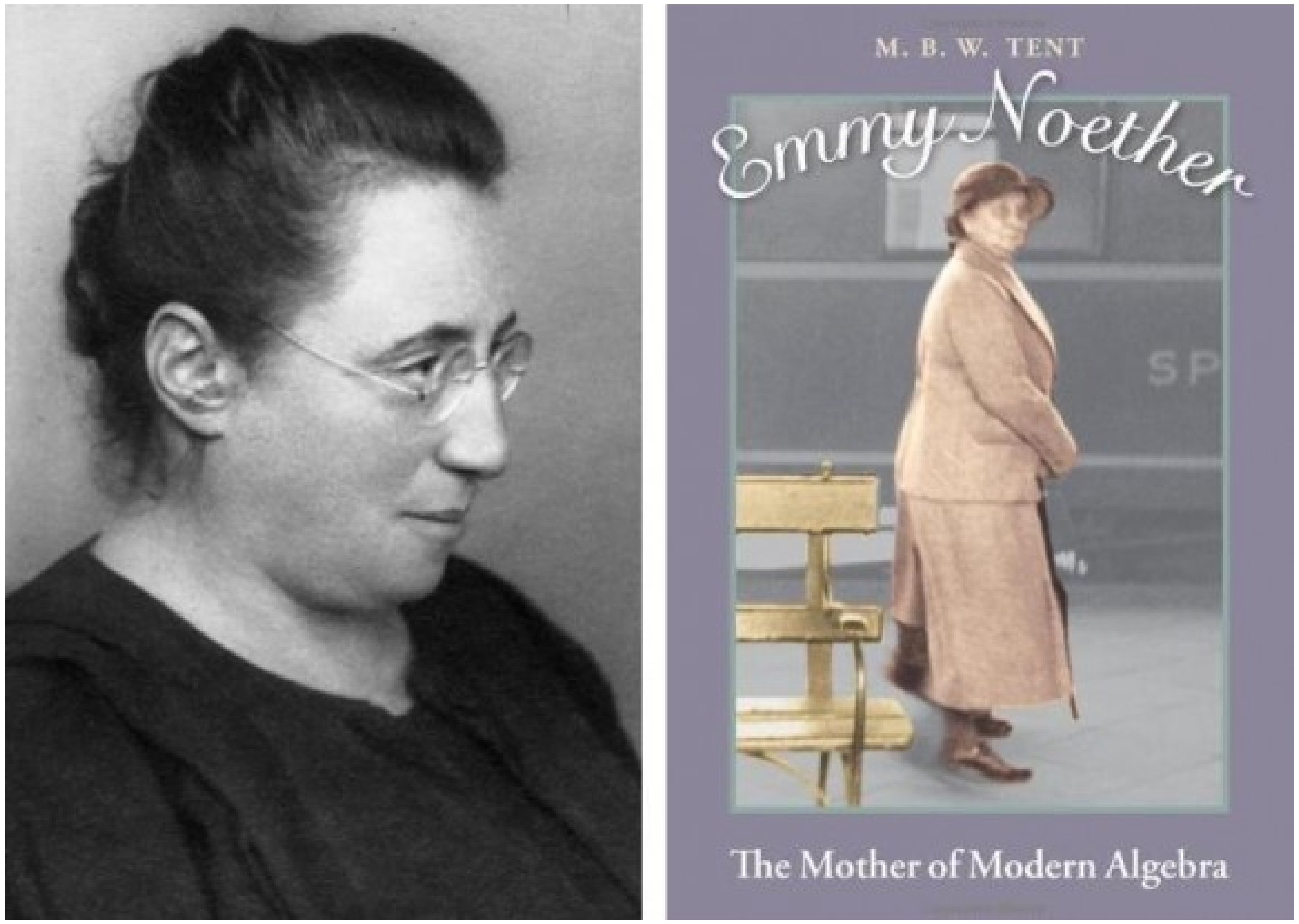
Step 2: Take for x an element which does not belong to intermediate subfields $K \supsetneq K' \supset k$. Such an element exists by Proposition 1, because there is a finite sets of K', and they have positive codimension in K considered as a vector space over k. **Then x is primitive,** because it generates a subfield which is equal to K. ■

Emmy Noether's postcard (1915)



Emmy Noether, a postcard sent to Ernst Fischer
April 10, 1915

Emmy Noether (1882-1935)



Emmy Noether (1882-1935)

Noether's normalization lemma (second version)

The following theorem was proven by Emmy Noether in 1926.

THEOREM: (Noether's normalization lemma, second version) Let $X \subset \mathbb{C}^n$ be an irreducible, normal affine subvariety, and k the transcendence degree of X (number of elements in the transcendence basis of $[k(X) : \mathbb{C}]$). Then there exists a variety $X_1 \subset \mathbb{C}^{k+1}$, given by a polynomial equation $P(t) = 0$, where $P(t)$ is a monic polynomial with coefficients in $\mathbb{C}[z_1, ..., z_k]$, such that **$X$ is isomorphic to the normalization of X_1.**

Proof. Step 1: Let $X \subset \mathbb{C}^n$, with coordinates $z_1, ..., z_n$, and $z_1, ..., z_k$ a transcendence basis in $k(X)$. Then a general linear combination $\tau := \sum_{i=1}^{k-1} a_{k+i} z_{k+i}$ is primitive in $[k(X) : k(z_1, ..., z_k)]$. Indeed, any proper subfield $K \subsetneq k(X)$ does not contain the k-subspace W generated by $z_{k+1}, ..., z_n$, because W generates K multiplicatively. There are only finitely many subfields K_i with $k(z_1, ..., z_k) \subset K_i \subsetneq k(X)$. Since $W \not\subset K_i$, one has $W \not\subset \bigcup K_i$ (Proposition 1). **Any element $\tau \in W \backslash \bigcup K_i$ is primitive.**

Noether's normalization lemma (2)

THEOREM: (Noether's normalization lemma, second version) Let $X \subset \mathbb{C}^n$ be an irreducible, normal affine subvariety, and k the transcendence degree of X (number of elements in the transcendence basis of $[k(X):\mathbb{C}]$). Then there exists a variety $X_1 \subset \mathbb{C}^{k+1}$, given by a polynomial equation $P(t)=0$, where $P(t)$ is a monic polynomial with coefficients in $\mathbb{C}[z_1,...,z_k]$, such that **X is isomorphic to the normalization of X_1.**

Step 2: Let $\Pi_{k+1}: \mathbb{C}^n \longrightarrow \mathbb{C}^{k+1}$ be the projection to the coordinates $z_1,...,z_k,\tau$, chosen in Step 1, and X_1 its image, that is, $X_1 = \operatorname{Spec}(B)$, where $B \subset \mathcal{O}_X$ is the subalgebra generated by $z_1,...,z_k,\tau$. After an appropriate linear change of coordinates, we can assume that $\Pi_{k+1}: X \longrightarrow X_1$ is finite (Corollary 1) and birational (Step 1). **Also, $\mathcal{O}_{X_1} = \mathbb{C}[z_1,...,z_k,t]/(P)$ where $P(z_1,...,z_k,t)$ is the monic polynomial constructed in Noether normalization lemma, version 1.**

Step 3: The projection $X \longrightarrow X_1$ is birational and finite, and X is normal. Therefore, X is the normalization of X_1. ■