Commutative Algebra

lecture 17: Krull dimension and algebraic dimension

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Algebraic dimension

DEFINITION: Let $A \supset \mathbb{C}[t_1, ..., t_n]$ be a ring without zero divisors. Assume that the extension $[k(A) : \mathbb{C}(t_1, ..., t_n)]$ is finite. Then n is called **the transcence degree** of A.

REMARK: Let M be an irreducible affine variety, and $\pi : M \longrightarrow \mathbb{C}^n$ the finite, dominant map constructed in Noether's normalization lemma. By construction, n **is equal to the transcendence degree of** $[k(M) : \mathbb{C}]$**.**

DEFINITION: Let M be an irreducible affine variety. **Algebraic dimension** $a(M)$ is the transcendence degree of $[k(M) : \mathbb{C}]$.

Divisors in $\mathbb{C}^n$

DEFINITION: Let X be an affine variety, and $f \in \mathcal{O}_X$ a regular function which does not vanish on any of irreducible components of X. The zero set of f is called **a principal divisor** on X. Its irreducible components are called **divisors** on X.

LEMMA: **Any divisor in $\mathbb{C}^n$ is principal.**

Proof: Let $I = (f) \subset \mathbb{C}[t_1, ..., t_n]$ be a principal ideal, $f = \prod_i f_i^{\alpha_i}$ be the prime decomposition of f, and $V_{(f)}$, $V_{(f_i)}$ be the corresponding zero sets. The ideals (f_i) are prine, because $\mathbb{C}[t_1, ..., t_n]$ Since $V_I = \bigcup V_{(f_i)}$, and all the ideals (f_i) are prime, the decomposition $V_I = \bigcup V_{(f_i)}$ coincides with the irreducible decomposition of $V_{(f)}$. We obtain that **any irreducible component of $V_{(f)}$ is principal.** ■

Algebraic dimension of a divisor

Proposition 1: **Let D be an irreducible divisor in $\mathbb{C}^n$. Then $a(D) = n-1$.**

Proof: Let $D = V_{(f)}$, where f is an irreducible polynomial (it exists by the previous lemma). **Then $f = 0$ gives an algebraic relation between the coordinate functions, hence the maximal number of algebraically independent coordinate functions on D is $n-1$.** Moreover, for each of the coordinates, say, t_n, such that f depends non-trivially on t_n, its image in $k(D)$ is a root of a polynomial with coefficients in $\mathbb{C}[t_1, ..., t_{n-1}]$. Therefore $k(D)$ is a finite extension of $\mathbb{C}(t_1, ..., t_{n-1})$.

■

Krull dimension

REMARK: Length of a chain $A_1 \subset A_2 \subset A_3 \subset ... \subset A_n$ is $n-1$, that is, the number of $\subset$ signs.

DEFINITION: Krull dimension of a ring A is the maximal possible length of a chain of prime ideals $0 \neq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 ... \subsetneq \mathfrak{p}_n \subsetneq A$

DEFINITION: Krull dimension of a variety X is the maximal possible length of a chain of non-empty, irreducible, distinct subvarieties $X_1 \subsetneq ... \subsetneq X_n$.

Today we are going to prove the following theorem

THEOREM: For any affine variety, **its algebraic dimension is equal to its Krull dimension.**

Wolfgang Krull (1899-1971)

Wolfgang Krull (1899-1971),
photo by Paul Halmos,
Seventh Brazilian Mathematics Colloquium
in Poços de Caldas, Brazil, July 1969,

Local rings and Nakayama lemma

DEFINITION: A ring A is called **local** if it has only one maximal ideal.

DEFINITION: Let $\mathfrak{p} \subset A$ be a prime ideal, and $S \subset A$ its complement. **Localization of A in $\mathfrak{p}$** is $A[S^{-1}]$.

CLAIM: Localization $A_\mathfrak{p}$ of A in $\mathfrak{p}$ is local.

Proof: Any $x \in A \backslash \mathfrak{p}$ is invertible, hence $\mathfrak{p}$ **is a maximal ideal, containing all ideals in A.** ■

THEOREM: (Nakayama's lemma for local rings)
Let A be a Noetherian local ring, $\mathfrak{p}$ its maximal ideal, and M a finitely generated A-module. Then $M \supsetneq \mathfrak{p}M$.

Proof: For any non-trivial ideal $\mathfrak{a} \subset A$, **Nakayama lemma claims that $\mathfrak{a}M = M$ implies that $(1+a)M = 0$,** for some $a \in \mathfrak{a}$. For any $a \in \mathfrak{p}$, $1 + a$ is invertible, hence $M = 0$. ■

Finite ring extensions and prime ideals: going down

DEFINITION: Let $B \supset A$ be a ring, which is finitely generated as an A-module. In this case, we say that **B is finite extension of A.**

Lemma 1: Let $B \supset A$ be a extension of a ring A without zero divisors, and $\mathfrak{q} \subset B$ a non-zero prime ideal. **Them $\mathfrak{p} := \mathfrak{q} \cap A$ is nonzero.**

Proof: Consider the ring $A_\mathfrak{p} = A[S^{-1}]$ localized in the set S of all $s \notin \mathfrak{p}$, and let $B_\mathfrak{p} := B[S^{-1}]$. Then $B_\mathfrak{p} \supset A_\mathfrak{p}$ is a finite $A_\mathfrak{p}$-module. If $\mathfrak{p} = 0$, $A_\mathfrak{p}$ is a field, and then $B_\mathfrak{p}$ is also a field as follows from the classification of semisimple Artinian algebras over a field. However, $B_\mathfrak{p}$ contains a non-trivial ideal $\mathfrak{q} \neq 0$, hence it cannot be a field. ■

Finite ring extensions and prime ideals: going up

Lemma 2: Let $B \supset A$ be a finite extension of a Noetherian ring A, and $\mathfrak{p} \subset A$ a prime ideal. **Then there exists finitely many prime ideals $\mathfrak{q} \subset B$ such that $\mathfrak{p} = \mathfrak{q} \cap A$.**

Proof. Step 1: As above, consider the ring $A_\mathfrak{p} = A[S^{-1}]$ localized in the set S all $s \notin \mathfrak{p}$. The kernel of the natural map $A \longrightarrow A_\mathfrak{p}/\mathfrak{p}$ is $\mathfrak{p}$. Indeed, the map $A/\mathfrak{p} \longrightarrow A_\mathfrak{p}/\mathfrak{p}$ has no kernel because $\mathfrak{p}$ is prime, and the kernel of $A \longrightarrow A/\mathfrak{p}$ is $\mathfrak{p}$.

Step 2: Let $B_\mathfrak{p} := B[S^{-1}]$. By Nakayama's lemma, $B_\mathfrak{p} \neq \mathfrak{p}B_\mathfrak{p}$. Then $B_\mathfrak{p}/\mathfrak{p} = B_\mathfrak{p} \otimes_A A_\mathfrak{p}/\mathfrak{p}$ is a non-zero, finite-dimensional ring over the field $A_\mathfrak{p}/\mathfrak{p}$. Let $\tilde{\mathfrak{q}}$ be any prime ideal in $B_\mathfrak{p}/\mathfrak{p}$ (there are finitely many prime ideals by classification of Artinian algebras), and let $\mathfrak{q}$ be the preimage of $\tilde{\mathfrak{q}}$ inder the natural map $B \longrightarrow B_\mathfrak{p}/\mathfrak{p}$. Then $\mathfrak{q}$ is prime, and $\mathfrak{q} \cap A$ is mapped to 0 under the natural map $A \longrightarrow A_\mathfrak{p}/\mathfrak{p}$, hence $\mathfrak{q} \cap A = \mathfrak{p}$ (Step 1). ■

Cohen-Seidenberg theorems

THEOREM: (Cohen-Seidenberg theorem)
Let $B \supset A$ be a finite Noetherian ring over A, and $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 ... \subsetneq \mathfrak{q}_n \subsetneq B$ be a chain of prime ideals. Denote by $\mathfrak{p}_i$ the ideal $\mathfrak{p}_i \cap A \subset A$; it is clearly prime. Then

(i) $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 ... \subsetneq \mathfrak{p}_n \subsetneq A$ **(distinct prime ideals remain distinct)**

(ii) Any chain of prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 ... \subsetneq \mathfrak{p}_n \subsetneq A$ **is obtained this way.**

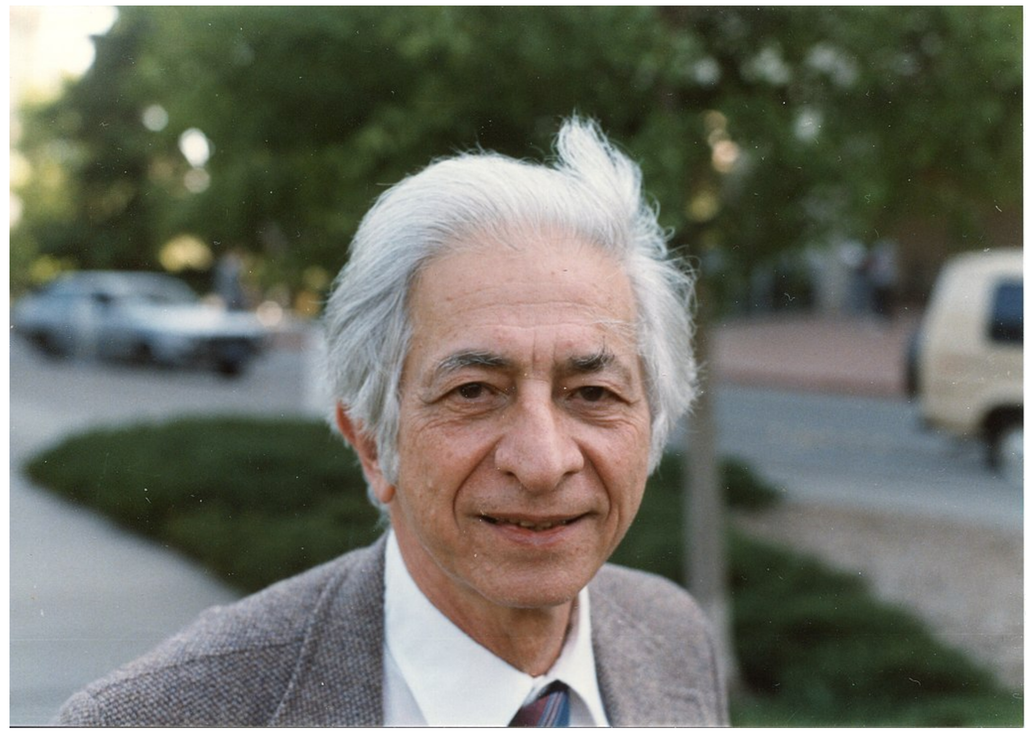
Proof of (i): Suppose that $\mathfrak{p}_i = \mathfrak{p}_{i-1}$. Replacing A by $A/\mathfrak{p}_{i-1}$ and B by $B/\mathfrak{q}_{i-1}$, we reduce the statement of (i) to Lemma 1.

Proof of (ii): Existence of $\mathfrak{q}_1$ follows from Lemma 2. Using induction, we may assume that $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 \subsetneq ... \subsetneq \mathfrak{q}_r$ is already chosen. To prove the induction step, we need to chose a prime ideal $\mathfrak{q}_{r+1}$ in $B/\mathfrak{q}_r$ such that $\mathfrak{q}_{r+1} \cap A/\mathfrak{q}_r = \mathfrak{p}_{r+1}$. This is again Lemma 2. ■

Irving S. Cohen and Abraham Seidenberg

Irving S. Cohen and Abraham Seidenberg, "Prime ideals and integral dependence", 1946, Bull. Amer. Math. Soc. 52 (4): 252-261

Irvin Sol Cohen
(1917-1955)



Abraham Seidenberg
(1916-1988)

Krull dimension is invariant under finite morphisms

COROLLARY: Let $X \longrightarrow Y$ be a finite, dominant morphism of irreducible affine varieties. **Then the Krull dimension of X is equal to Krull dimension of Y.**

Proof: Any chain of prime ideals in $\mathcal{O}_Y \subset \mathcal{O}_X$ can be lifted to $\mathcal{O}_X$ by Cohen-Seidenberg; any chain of distinct prime ideals in $\mathcal{O}_X$ intersected with $\mathcal{O}_Y$ gives a chain of distinct prime ideals in $\mathcal{O}_Y$, again by Cohen-Seidenberg. ■

The Krull dimension and the algebraic dimension

THEOREM: For any affine variety X, **its algebraic dimension $a(X)$ is equal to its Krull dimension $\dim X$.**

Proof. Step 1: By Noether's lemma, there exists a finite, dominant map $X \longrightarrow \mathbb{C}^d$, where $d = a(X)$ is the algebraic dimension. It remains to show that the Krull dimension of $\mathbb{C}^d$ is d.

Step 2: Any prime ideal which is not principal contains an irreducible polynomial, hence it contains a principal prime ideal. This implies that **in any maximal chain $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 ...$ of prime ideals in $\mathcal{O}_{\mathbb{C}^n}$, the ideal $\mathfrak{p}_1$ is principal.**

Step 3: We obtained that $\dim \mathbb{C}^n = 1 + \dim D$, where D is an irreducible divisor. Using induction in $\dim X$, we may assume that $\dim Y = a(Y)$ for any affine variety of algebraic dimension $< n$. Then $\dim D = a(D) = n - 1$ (Proposition 1), giving $\dim \mathbb{C}^n = \dim D + 1 = n$. ■

The proof of Hilbert Nullstellensatz

Using Noether's normalization lemma, we can prove the result which can be used to deduce Hilbert Nullstellensatz. In Lecture 1, we proved this theorem using the set theory.

THEOREM: Let F be a finitely generated algebra over an algebraically closed field $\overline{k}$. **Assume that F is a field. Then $F = \overline{k}$.**

Proof: Noether normalization lemma implies that there exists an embedding $\overline{k}[t_1, t_2, ..., t_n] \hookrightarrow F$ such that F is finite generated as $\overline{k}[t_1, t_2, ..., t_n]$-module. By Lemma 1, this implies that **every non-zero prime ideal in $\overline{k}[t_1, t_2, ..., t_n]$ is extended to a non-zero prime ideal in F.** However, F is a field, haence it has no non-zero ideals, and $n = 0$. Therefore, F is a finite extension of $\overline{k}$. Since $\overline{k}$ is algebraically closed, any finite extension of $\overline{k}$ is trivial. ■