Commutative Algebra

lecture 21: Smooth points in algebraic varieties

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http://verbit.ru/IMPA/CA-2022/

IMPA, sala 232

February 21, 2022

Topological manifolds

REMARK: Manifolds can be smooth (of a given "differentiability class"), real analytic, or topological (continuous).

DEFINITION: **Topological manifold** is a topological space which is locally homeomorphic to an open ball in $\mathbb{R}^n$.

EXERCISE: **Show that a group of homeomorphisms acts on a connected manifold transitively.**

DEFINITION: Such a topological space is called **homogeneous**.

Open problem: (Busemann)
Characterize manifolds among other homogeneous topological spaces.

Banach fixed point theorem

LEMMA: (Banach fixed point theorem/"contraction principle")
Let $U \subset \mathbb{R}^n$ be a closed subset, and $f: U \longrightarrow U$ a map which satisfies $|f(x) - f(y)| < k|x - y|$, where $k < 1$ is a real number (such a map is called **"contraction"**). **Then f has a fixed point, which is unique.**

Proof. Step 1: Uniqueness is clear because for two fixed points x_1 and x_2 $|f(x_1) - f(x_2)| = |x_1 - x_2| < k|x_1 - x_2|$.

Step 2: Existence follows because the sequence $x_0 = x, x_1 = f(x), x_2 = f(f(x)), ...$ satisfies $|x_i - x_{i+1}| \leqslant k|x_{i-1} - x_i|$ which gives $|x_n - x_{n+1}| < k^n a$, where $a = |x - f(x)|$. Then $|x_n - x_{n+m}| < \sum_{i=0}^m k^{n+i} a \leqslant k^n \frac{1}{1-k} a$, hence $\{x_i\}$ is a Cauchy sequence, and converges to a limit y, which is unique.

Step 3: $f(y)$ is a limit of a sequence $f(x_0), f(x_1), ... f(x_i), ...$ which gives $y = f(y)$. ■

EXERCISE: Find a counterexample to this statement when U is open and not closed.

Differentiable maps

DEFINITION: Let $U, V \subset \mathbb{R}^n$ be open subsets. **An affine map** is a sum of linear map α and a constant map. Its **linear part** is α.

DEFINITION: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets. A map $f: U \longrightarrow V$ is called **differentiable** if it can be approximated by an affine one at any point: that is, for any $x \in U$, there exists an affine map $\varphi_x: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that

$$\lim_{x_1 \to x} \frac{|f(x_1) - \varphi(x_1)|}{|x - x_1|} = 0$$

DEFINITION: **Differential**, or **derivative** of a differentiable map $f: U \longrightarrow V$ is the linear part of φ.

DEFINITION: **Diffeomorphism** is a differentiable map f which is invertible, and such that f^{-1} is also differentiable. A map $f: U \longrightarrow V$ is a **local diffeomorphism** if each point $x \in U$ has an open neighbourhood $U_1 \ni x$ such that $f: U_1 \longrightarrow f(U_1)$ is a diffeomorphism.

REMARK: **Chain rule** says that a composition of two differentiable functions is differentiable, and its differential is composition of their differentials.

REMARK: Chain rule implies that **differential of a diffeomorphism is invertible.** Converse is also true (in a sense):

Inverse function theorem

THEOREM: Let $U, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a differentiable map. Suppose that the differential of f is everywhere invertible. **Then f is locally a diffeomorphism.**

Proof. Step 1: Let $x \in U$. Without restricting generality, we may assume that $x = 0$, $U = B_r(0)$ is an open ball of radius r, and **in U one has $\frac{|f(x_1)-\varphi(x_1)|}{|x-x_1|} < 1/2$.** Replacing f with $-f \circ (D_0 f)^{-1}$, where $D_0 f$ is differential of f in 0, **we may assume also that $D_0 f = -\operatorname{Id}$.**

Step 2: In these assumptions, $|f(x)+x| < 1/2|x|$, hence $\psi_s(x) := f(x)+x-s$ is a contraction. This map maps $\overline{B}_{r/2}(0)$ to itself when $s < r/4$. By Banach fixed point theorem, **$\psi_s(x) = x$ has a unique fixed point x_s, which is obtained as a solution of the equation $f(x)+x-s = x$, or, equivalently, $f(x) = s$.** Denote the map $s \longrightarrow x_s$ by g.

Step 3: By construction, $fg = \operatorname{Id}$. Applying the chain rule again, we find that g is also differentiable. ■

REMARK: Usually, diffeomorphisms are assumed **smooth** (infinitely differentiable). **A smooth version of this result is left as an exercise.**

Critical points and critical values

DEFINITION: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a smooth function. A point $x \in U$ is a **critical point** of f if the differential $D_x f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is not surjective. **Critical value** is an image of a critical point. **Regular value** is a point of V which is not a critical value.

THEOREM: (Sard's lemma)
The set of critical values of f is of measure 0 in V.

REMARK: We leave this theorem without a proof. We won't use it.

DEFINITION: A subset $M \subset \mathbb{R}^n$ is **an m-dimensional smooth submanifold** if for each $x \in M$ there exists an open in $\mathbb{R}^n$ neighbourhood $U \ni x$ and a diffeomorphism from U to an open ball $B \subset \mathbb{R}^n$ which maps $U \cap M$ to an intersection $B \cap R^m$ of B and an m-dimensional linear subspace.

REMARK: Clearly, **a smooth submanifold is a (topological) manifold.**

THEOREM: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, $f: U \longrightarrow V$ a smooth function, and $y \in V$ a regular value of f. **Then $f^{-1}(y)$ is a smooth submanifold of U.**

Preimage of a regular value

THEOREM: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets, $f: U \longrightarrow V$ a smooth function, and $y \in V$ a regular value of f. **Then $f^{-1}(y)$ is a smooth submanifold of U.**

Proof: Let $x \in U$ be a point in $f^{-1}(y)$. It suffices to prove that x has a neighbourhood diffeomorphic to an open ball B, such that $f^{-1}(y)$ corresponds to a linear subspace in B. Without restricting generality, we may assume that $y = 0$ and $x = 0$.

The differential $D_0 f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is surjective. Let $L := \ker D_0 f$, and let $A: \mathbb{R}^n \longrightarrow L$ be any map which acts on L as identity. Then $D_0 f \oplus A: \mathbb{R}^n \longrightarrow \mathbb{R}^m \oplus L$ is an isomorphism of vector spaces. **Therefore, $\Psi: f \oplus A$ mapping x_1 to $f(x_1) \oplus A(x_1)$ is a diffeomorphism in a neighbourhood of x.** However, $f^{-1}(0) = \Psi^{-1}(0 \oplus L)$. We have constructed a diffeomorphism of a neighbourhood of x with an open ball mapping $f^{-1}0)$ to $0 \oplus L$. ■

Regular maps and solutions of a set of equations

COROLLARY: Let $f_1, ..., f_m$ be smooth functions on $U \subset \mathbb{R}^n$ such that the differentials df_i are linearly independent everywhere. **Then the set of solutions of equations $f_1(z) = f_2(z) = ... = f_m(z) = 0$ is a smooth $(n-m)$-dimensional submanifold in U.**

DEFINITION: **Smooth hypersurface** is a closed codimension 1 submanifold.

EXERCISE: Prove that **a smooth hypersurface in U is always obtained as a solution of an equation $f(z) = 0$,** where 0 is a regular value of a function $f : U \longrightarrow \mathbb{R}$.

Abstract manifolds: charts and atlases

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a **refinement** of a cover $\{U_i\}$ if every V_i is contained in some U_i.

REMARK: Any two covers $\{U_i\}$, $\{V_i\}$ of a topological space admit a common refinement $\{U_i \cap V_j\}$.

DEFINITION: Let M be a topological manifold. A cover $\{U_i\}$ of M is an **atlas** if for every U_i, we have a map $\varphi_i : U_i \to \mathbb{R}^n$ giving a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. In this case, one defines the **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

DEFINITION: A function $\mathbb{R} \longrightarrow \mathbb{R}$ is **of differentiability class C^i** if it is i times differentiable, and its i-th derivative is continuous. A map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **of differentiability class C^i** if all its coordinate components are. A **smooth function/map** is a function/map of class $C^\infty = \bigcap C^i$.

DEFINITION: An atlas is **smooth** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), **smooth of class C^i** if all transition functions are of differentiability class C^i, and **real analytic** if all transition maps admit a Taylor expansion at each point.

Covering maps

DEFINITION: Let $\varphi: \tilde{M} \longrightarrow M$ be a continuous map of manifolds (or CW complexes). We say that φ is **a covering** if φ is locally a homeomorphism, and for any $x \in M$ there exists a neighbourhood $U \ni x$ such that is a disconnected union of several manifolds U_i such that the restriction $\varphi\big|_{U_i}$ is a homeomorphism.

DEFINITION: Let Γ be a discrete group continuously acting on a locally compact topological space M. This action is called **properly discontinuous** if the space of orbits of Γ is Hausdorff.

THEOREM: Let Γ be a discrete group acting on M freely and properly discontinuously. **Then $M \longrightarrow M/\Gamma$ is a covering.**

This result is left as an exercise.

Finite coverings

EXAMPLE: A map $x \longrightarrow nx$ in a circle S^1 is a covering.

EXAMPLE: For any non-degenerate integer matrix $A \in \operatorname{End}(\mathbb{Z}^n)$, the corresponding map of a torus T^n is a covering.

CLAIM: Let $\varphi: \tilde{M} \longrightarrow M$ be a covering, with M connected. **Then the number of preimages $|\varphi^{-1}(m)|$ is constant in M.**

Proof: Since $\varphi^{-1}(U)$ is a disconnected union of several copies of U, this number is a locally constant function of m. ■

DEFINITION: Let $\varphi: \tilde{M} \longrightarrow M$ be a covering, with M connected. The number $|\varphi^{-1}(m)|$ is called **degree** of a map φ.

CLAIM: Any covering $\varphi: \tilde{M} \longrightarrow M$ with $\tilde{M}$ compact **has finite degree.**

Proof: Take U in such a way that $\varphi^{-1}(U)$ is a disconnected union of several copies of U, and let $x \in U$. Then $\varphi^{-1}(x)$ is discrete, and since $\tilde{M}$ is compact, any discrete subset of $\tilde{M}$ is finite. ■

Finite projection maps (reminder)

PROPOSITION: Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, z_i coordinates on $\mathbb{C}^n$, and $z_1, ..., z_k$ transcendence basis on $k(X)$. Then, for all $\lambda_1, ..., \lambda_k$ outside of the zero-set of a certain non-zero homogeneous polynomial, **the function $z_n \in \mathcal{O}_X$ is a root of a monic polynomial in the variables $z'_1, ..., z'_k$, where $z'_i := z_i + \lambda_i z_n$.**

Proof: Lecture 15. ∎

Corollary 1: (Noether's normalization lemma, first version)
Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, z_i coordinates on $\mathbb{C}^n$, and $z_1, ..., z_k$ transcendence basis on $k(X)$. Then **there exists a linear coordinate change $z'_i := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$, such that the projection $\Pi_k : X \longrightarrow C^k$ to the first k arguments is a finite, dominant morphism.**

Proof: Previous proposition shows that the projection $P_n : X \longrightarrow \mathbb{C}^{n-1}$ is finite onto its image X_1 (after some linear adjustment). Using induction by n, we can assume that $P_k : X_1 \longrightarrow \mathbb{C}^k$ is also finite, hence the composition map is finite **(composition of finite morphisms is always finite, as we have seen).** ∎

Multi-valued functions

DEFINITION: Define **a complex manifold** as a manifold equipped with an atlas $\{U_i\}$, with each open subset $U_i \subset M$ identified with an open ball in $\mathbb{C}^n$, and complex analytic transition functions.

DEFINITION: Define **a complex variety** as a subvariety $Z \subset M$ in a complex manifold given by a collection of complex analytic equation.

DEFINITION: **Multi-valued function** on M is a closed, irreducible complex subvariety $Z \subset M \times \mathbb{C}$ such that the projection $Z \longrightarrow M$ is locally a diffeomorphism outside of a closed, nowhere dense subset in Z. The set Z is called **the graph of the multi-valued function**.

EXAMPLE: **Logarithm is a multi-valued function** on $\mathbb{C}$. Indeed, let Z be the graph of exponent $y = e^x$ in $\mathbb{C}^2$. The projection to x expresses all branches of logarithm $x = \log y$ as functions of y.

EXAMPLE: **$y \longrightarrow \sqrt{y}$ is a multi-valued function.** Indeed, the graph of $y = x^2$ projected to x gives both branches of $\sqrt{y}$.

Multi-valued functions and branched covers

THEOREM: Consider a subvariety $Z \subset \mathbb{C}^{n+1}$ given by a monic polynomial equation $P(t) = 0$, with $P(t) \in \mathcal{O}_{\mathbb{C}^n}[t]$. Assume that $P(t)$ is irreducible. **Then Z is a graph of a multi-valued function.** Moreover, the projection of Z to $\mathbb{C}^n$ (to the first n coordinates) is a diffeomorphism at (z,t) if and only if $P'(z) \neq 0$.

Proof. Step 1: We shall represent points of $\mathbb{C}^{n+1}$ by pairs (z,t), with $z = (z_1, ..., z_n)$. Let $\pi: Z \longrightarrow \mathbb{C}^n$ be the standard projection along t. By the inverse function theorem, **$Z \subset \mathbb{C}^{n+1}$ is a smooth submanifold in a neighbourhood of any point $(z,t) \in Z$, with $z \in \mathbb{C}^n$ whenever the differential $dP: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ is surjective** (non-zero) at (z,t).

Step 2: This implies that Z is smooth outside of an algebraic subset of all $(z,t) \in \mathbb{C}^{n+1}$ such that $dP(z,t)(\xi, \tau) = 0$, for all $\xi \in \langle d/dz_1, ..., d/dz_n\rangle$, and $\tau \in \langle d/dt\rangle$. Let $P(z,t) = t^k + \sum_{i=0}^{k-1} t^i a_i(z)$. Then

$$dP(z,t) = kt^{k-1}dt + \sum_{i=0}^{k-1} t^i da_i(z) + \sum_{i=0}^{k-1} it^{i-1} a_i(z)dt.$$

For $|t| \gg 0$, the leading term $nt^{k-1}dt + t^{k-1}da_{k-1}(z)$ dominates the rest, and it is non-zero, because its dt component is non-zero.

Multi-valued functions and branched covers (2)

THEOREM: Consider a subvariety $Z \subset \mathbb{C}^{n+1}$ given by a monic polynomial equation $P(t) = 0$, with $P(t) \in \mathcal{O}_{\mathbb{C}^n}[t]$. Assume that $P(t)$ is irreducible. **Then Z is a graph of a multi-valued function.** Moreover, Z is smooth, and the projection of Z to $\mathbb{C}^n$ (to the first n coordinates) is a diffeomorphism at (z, t) if and only if $P'(z) \neq 0$.

Step 3: Let $z \in Z$ be a smooth point, and $(\xi, \tau) \in T_z Z$. Then $\pi : W \longrightarrow \mathbb{C}^n$ is invertible whenever W does not contain a vector $(0, \tau)$, equivalently, when $dP(z,t)(0, \tau) \neq 0$. **This is equivalent to $\frac{dP(z,t)}{dt} \neq 0$.**

Step 4: Let $P'(z,t) := \frac{dP(z,t)}{dt}$. To prove that Z defines a multi-valued function, it remains to show that $P'(z,t)$ is not identically zero on Z. Since $P(z,t)$ is irreducible, and $\mathcal{O}_{\mathbb{C}^{n+1}}$ is factorial, the ring $\frac{\mathcal{O}_{\mathbb{C}^{n+1}}}{(P)}$ has no zero divisors. Then Hilbert Nullstellensatz would imply that any function $f \in \mathcal{O}_{\mathbb{C}^{n+1}}$ which vanishes on Z is divisible by $P(z,t)$. Then $P'(z,t)$ does not vanish on Z, because it is polynomial of smaller degree. ∎

Symmetric polynomials

DEFINITION: Symmetric polynomial $P(z_1, ..., z_n) \in \mathbb{C}[z_1, ..., z_n]$ is a polynomial which is invariant with respect to the symmetric group Σ_n acting on $\mathbb{C}[z_1, ..., z_n]$ in a usual way.

DEFINITION: Consider the polynomial $P(z_1, ..., z_n, t) := \prod_{i=1}^n (t + z_i) = \sum e_i t^i$, with $e_i \in \mathbb{C}[z_1, ..., z_n]$. Then e_i are called **elementary symmetric polynomials** on $z_1, ..., z_n$.

THEOREM: Every symmetric polynomial on $z_1, ..., z_n$ can be polynomially expressed through the elementary symmetric polynomials.

Proof: Left as an exercise. ■

Discriminant of a polynomial

DEFINITION: Consider the symmetric polynomial $\prod_{i\neq j}(z_i - z_j)$. **Discriminant** of the polynomial $P(z_1, ..., z_n, t) := \prod_{i=1}^n (t - z_i)$ is $\prod_{i\neq j}(z_i - z_j)$ considered as a polynomial of its coefficients.

EXAMPLE: Discriminant of the quadratic polynomial $t^2 + bt + c$ is $b^2 - 4c$.

EXAMPLE: Discriminant of the cubic polynomial $t^3 + bt^2 + ct + d$ is $b^2c^2 - 4c^3 - 4b^3d - 27d^2 + 18bcd$.

CLAIM: A polynomial has no multiple roots if and only if its discriminant is non-zero. ■

Corollary 1: Let $P(t) \in k[t]$ be a polynomial over an algebraically closed field, and D its discriminant. **Then the derivative $P'(t)$ does not vanish on all roots of t if and only if $D \neq 0$.**

Proof: Let α be a root of P. Then $P(t) = P_\alpha(t)(t - \alpha)$, and $P'(t) = P'_\alpha(t)(t - \alpha) + P_\alpha(t)$. Therefore, $P'(\alpha) = 0$ if and only if $P_\alpha(t) = 0$. This is equivalent to $P(t)$ being divisible by $(t - \alpha)^2$. ■

Discriminant and ramified coverings

THEOREM: Consider a subvariety $Z \subset \mathbb{C}^{n+1}$ given by an irreducible monic polynomial equation $P(t) = 0$ of degree k, with $P(t) \in \mathbb{C}[z_1, ..., z_n][t]$, and let $\pi : Z \longrightarrow \mathbb{C}^n$ be the projection to the coordinates $z_1, ..., z_n$. Assume that $P(t)$ is irreducible. Denote by $D(z)$ the discriminant of $P(t)$, considered as a polynomial function on $(z_1, ..., z_n)$, and let $U \subset \mathbb{C}^n$ be the set of all $z \in \mathbb{C}^n$ such that $D(z) \neq 0$. **Then the intersection $Z \cap \pi^{-1}(U)$ is smooth and the projection $\pi : Z \cap \pi^{-1}(U) \longrightarrow U$ is a d-sheeted covering.**

Proof. Step 1: Since $P(t)$ is irreducible, it has no common divisors with $P'(t)$. Since the polynomial ring is factorial, irreducibility implies that the ideal $(P(t))$ is prime, hence $P'(t)$ does not identically vanish on Z. Therefore U is open and nowhere dense. By Corollary 1, for any $z \in U$, the polynomials $P(z,t)$ and $P'(z,t)$ have no common roots. Therefore, $dP(z,t) \neq 0$ on $Z \cap \pi^{-1}(U)$, and **the set $Z = \{(z,t) \mid P(z,t) = 0\}$ is smooth outside of zeros of $D(z)$.**

Step 2: Let $z \in Z \cap \pi^{-1}(U)$ and $(\xi, \tau) \in T_{(z,t)}Z$. Then $\pi : T_{(z,t)}Z \longrightarrow \mathbb{C}^n$ is invertible whenever $T_{(z,t)}$ does not contain a vector $(0, \tau)$, equivalently, when $dP(z,t)(0,\tau) \neq 0$. **This is equivalent to $\frac{dP(z,t)}{dt} \neq 0$.**

Step 3: The map $\pi : Z \cap \pi^{-1}(U) \longrightarrow U$ is locally a diffeomorphism, and each point has precisely d preimages. **Then it is a covering (prove it).** ■

Every algebraic variety is a ramified cover

Comparing this with the Noether normalization lemma, we obtain the following theorem.

COROLLARY: Let X be an algebraic variety. Then there exists a birational, finite map $X \longrightarrow Z$, a divizor $D \subset Z$, and a divisor $D_1 \subset \mathbb{C}^n$, **such that $Z \backslash D$ is a d-sheeted covering of $\mathbb{C}^n \backslash D_1$.** ■

COROLLARY: Every algebraic variety X over $\mathbb{C}$ has a smooth point. Moreover, **non-smooth points of X are contained in a proper algebraic subvariety of X.**

Proof: Indeed, **every birational map is an isomorphism outside of a divisor.** ■