Commutative Algebra

lecture 23: Regular rings

Misha Verbitsky

http://verbit.ru/IMPA/CA-2022/

IMPA, sala 232

February 25, 2022

Associated graded ring (reminder)

DEFINITION: A **multiplicative filtration** on a ring A is a sequence $A = F_0 \supset F_1 \supset ...$ such that all F_i are closed under multiplication and satisfy $F_iF_j \subset F_{i+j}$. A ring equipped with a multiplicative filtration is called **filtered**. An **associated graded quotient** of a filtered ring is $\oplus_{i=0}^\infty A^i$, where $A^i = F_i/F_{i+1}$.

CLAIM: $A = F_0 \supset F_1 \supset F_2 \supset ...$ be a filtered ring, $a_1, a_2 \in F_i$ and $b_1, b_2 \in F_j$. Assume that $a_1 = a_2 \mod F_{i+1}$ and $b_1 = b_2 \mod F_{j+1}$. **Then** $a_1b_1 = a_2b_2 \mod F_{i+1}F_j + F_iF_{j+1}$.

REMARK: Since $F_{i+1}F_j + F_iF_{j+1} \subset F_{i+j+1}$, the product of $a \in F_i/F_{i+1}$ and $b \in F_j/F_{j+1}$ is well defined as an element of F_{i+j}/F_{i+j+1}. Therefore, **the associated graded ring $\oplus_{i=0}^\infty A^i$ is equipped with a natural ring structure.** This ring is called **the associated graded ring**.

EXAMPLE: Let $I \subset A$ be an ideal. Then $A \supset I \supset I^2 \supset ...$ is a multiplicative filtration. The corresponding associated graded ring $A^* := \bigoplus_{i=0}^\infty \frac{I^i}{I^{i+1}}$ is called **the associated graded ring of the ideal I.**

Hilbert polynomial of a graded module (reminder)

DEFINITION: Let M^* be a finitely generated graded module over a graded ring A^*, with $A^0 = k$ a field. The **Hilbert function** of M^* is $h_M(n) := \dim_k M^n$.

THEOREM 1: Let M^* be a finitely generated graded module over a graded ring A^* generated by a finite-dimensional space A^1, with $A^0 = k$ a field. **Then the Hilbert function of $h_M(n)$ is polynomial for n sufficiently big.**

THEOREM 2: Let R be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, $A := \bigoplus_{i=0}^{\infty} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}$ its associated graded ring, and $h_A(n) := \dim A^n$. **Then the Hilbert function $h_A(n)$ is a polynomial for $n \gg 0$.** Moreover, **its degree d is equal to the Krull dimension of R.**

COROLLARY: The Krull dimension of a local Noetherian ring R **is equal to the Krull dimension of its associated graded ring.**

Regular rings

THEOREM 3: Let R be a Noetherian local ring of Krull dimension d, $\mathfrak{m}$ its maximal ideal, and $k := R/\mathfrak{m}$ its residue field. **Then** $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geqslant d$**.** Moreover, **the following are equivalent:**
(i) The associated graded ring of R is isomorphic to $k[t_1, ..., t_d]$;
(ii) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$;
(iii) the ideal $\mathfrak{m}$ can be generated by d elements.

Proof. Step 1: The implication (i) $\Rightarrow$ (ii) is obvious, and (ii) $\Rightarrow$ (iii) follows from Nakayama lemma, which also implies that the associated graded ring A^* of R is generated by A_1. Since A^* is a quotient of $k[A^1]$, and $\dim k[A^1] = \dim A^1$, we obtain $\dim_k \mathfrak{m}/\mathfrak{m}^2) = \dim k[A^1] \geqslant \dim A^* = \dim R = d$. **It remains to prove (iii) $\Rightarrow$ (i).**

Step 2: Let $\mu: \ k[A^1] \longrightarrow A^*$ be a natural surjective map, and $I := \ker \mu$. Let $t_1, ..., t_d$ be the basis in A^1; then $k[A^1] = k[t_1, ..., t_d]$. If I is non-empty, there is an algebraic relation between $t_i \in A^1$, which implies $d = \dim \operatorname{tr} A^* < \dim \operatorname{tr} k[A^1] = \dim_k A^1 = d$. **Therefore, $I = 0$ and μ is an isomorphism.** ∎

DEFINITION: A Noetherian local ring is called **regular** if (i)-(iii) holds.

Integral closure of a Noetherian ring

DEFINITION: An affine variety $X = \operatorname{Spec}(A)$ is called **smooth** if all local rings of maximal ideals in A are regular.

CLAIM: Discrete valuation rings are regular.

Proof: Indeed, the maximal ideal of the discrete valuation ring R is principal, hence $\mathfrak{m}/\mathfrak{m}^2 = R/\mathfrak{m} = k$. ■

COROLLARY: Let R be a ring of Krull dimension 1 and without zero divisors. **Then R is Dedekind if and only if it is smooth.** ■

LEMMA 1: Let R be a Noetherian ring without zero divisors, and $x \in k(R)$ an element in its fraction field. Then **x is integral over R if and only if there is $y \in R$ such that $yx^n \in R$ for all $n > 0$.**

Proof: Let $S = R[x] \subset k(X)$ be the subring generated by x. If $x = ab$, with $a, b \in R$, we have $b^n S \subset R$, hence $b^n x^m \in R$ for all $m \in \mathbb{Z}^{\geqslant 0}$.

Conversely, if $yx^n \in R$, then $aS \subset R$; the R-module aS is finitely generated, because R is Noetherian. ■

Regular rings are integrally closed

PROPOSITION: Let R be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, and $\operatorname{Gr}(R)$ the associated graded ring. Suppose that $\operatorname{Gr}(R)$ contains no zero divisors. **Then R has no zero divisors.** Moreover, **if $\operatorname{Gr}(R)$ is integrally closed then R is also integrally closed**.

Proof. Step 1: The first statement follows from Krull theorem, which gives $\bigcap_k \mathfrak{m}^k = 0$. If a product of two elements $a \in \mathfrak{m}^k \backslash \mathfrak{m}^{k+1}$ and $b \in \mathfrak{m}^l \backslash \mathfrak{m}^{l+1}$ vanishes, the product of these elements modulo $\mathfrak{m}^{k+1}$, $\mathfrak{m}^{l+1}$ vanishes in $\operatorname{Gr}(R)$. Further on, for any $a \in \mathfrak{m}^k \backslash \mathfrak{m}^{k+1}$, **we denote the class of $a \mod \mathfrak{m}^{k+1}$ by $\underline{a} \in \operatorname{Gr}(R)$**.

Step 2: Let $x = a/b$ be integral over R, with $a, b \in R$. Let $R_1 := R/(b)$. By Nakayama lemma, $\bigcap_k \mathfrak{m}^k R_1 = 0$, hence $\bigcap_k (\mathfrak{m}^k + (b)) = (b)$. To show that $a \in (b)$ **it would suffice to prove that $a \in \mathfrak{m}^k + (b)$ for all k.**

Step 3: We prove $a \in \mathfrak{m}^k + (b)$ using induction in k. Assume that $a \in \mathfrak{m}^{k-1} + (b) = cb + d$, where $c \in \mathfrak{m}^{k-1}$ and $d \in \mathfrak{m}^k \backslash \mathfrak{m}^{k+1}$. For some $y \in R$, and for all $n \in \mathbb{Z}^{>0}$ we have $y\frac{d^n}{b^n} \in R$ (Lemma 1). This brings $\underline{y}\frac{\underline{d}^n}{\underline{b}^n} \in R$. Since $\operatorname{Gr}(R)$ is integrally closed, this implies that $\frac{\underline{d}}{\underline{b}} \in \operatorname{Gr}(R)$, hence $d = bz \mod \mathfrak{m}^k$, and $a \in \mathfrak{m}^k + (b)$. ∎

COROLLARY: Regular rings are integrally closed. ∎

Kähler differentials

DEFINITION: Let R be a ring over a field k, and V an R-module. A k-linear map $D: R \longrightarrow V$ is called **a derivation** if it satisfies **the Leibnitz identity** $D(ab) = aD(b) + bD(a)$. The space of derivations from R to V is denoted $\operatorname{Der}_k(R, V)$.

REMARK: $\operatorname{Der}_k(R, V)$ **is an R-module**, with a natural R-action.

DEFINITION: Let R be a ring over a field k. Define an R-module $\Omega^1_k R$ **(the module of Kähler differentials)** with the following generators and relations.

* **Generators** of $\Omega^1_k R$ are indexed by elements of R; for each $a \in R$, the corresponding generator of $\Omega^1_k R$ is denoted da.

* **Relations** in $\Omega^1_k R$ are generated by expressions $d(ab) = adb + bda$, for all $a, b \in R$, and $d\lambda = 0$ for each $\lambda \in k$.

EXERCISE: Prove that the map $d: R \longrightarrow \Omega^1_k R$ mapping a to da **is a derivation**.

Universal property of Kähler differentials

CLAIM: Let V be an R-module, and $D \in \operatorname{Der}_k(R, V)$ a derivation. Then **there exists a unique R-module homomorphism $\varphi_D : \Omega^1_k R \longrightarrow V$ mapping bda to $bD(a)$.**

REMARK: Consider a category $\mathcal{C}$ of R-modules equipped with a derivation $(V, D : R \longrightarrow V)$, and define morphisms in $\mathcal{C}$ as morphisms of R-modules which commute with the derivation map. Then **$\Omega^1 R$ is an initial object in this category.** This is called **the universal property of the module of Kähler differentials**.

CLAIM: $\operatorname{Der}_k(R, V) = \operatorname{Hom}_R(\Omega^1 R, V)$.

Proof: A composition of a derivation $R \xrightarrow{d} \Omega^1 R$ and an R-module homomorphism $\Omega^1 R \longrightarrow V$ lies in $\operatorname{Der}_k(R, V)$. On the other hand, **any derivation $\xi \in \operatorname{Der}_k(R, V)$ is obtained this way,** by the universal property. ■

COROLLARY: $\operatorname{Der}_k(R) = (\Omega^1 R)^*$, where $V^* := \operatorname{Hom}(V, R)$.

EXERCISE: Prove that **$\Omega^1(\mathbb{C}[t_1, ..., t_n])$ is a free $\mathbb{C}[t_1, ..., t_n]$-module generated by $dt_1, ..., dt_n$.**

Exact sequence of Kähler differentials

Let $\delta: I \longrightarrow \Omega^1 A$ take x to dx. Since $d(xy) = yd(x) + xd(y)$, $\delta(I^2) = 0$ modulo $I\Omega^1 A$. Therefore, **the map $I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B$ is well defined.**

CLAIM: Let $\varphi: A \longrightarrow B$ be a surjective ring homomorphism, and I its kernel. **Then the following sequence of B-modules is exact: $I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B \longrightarrow 0.$**

Proof: The composition $\delta \circ \varphi$ clearly vanishes, and $\Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B$ is clearly surjective. The kernel of $\Omega^1 A \longrightarrow \Omega^1 B$ is generated by $I\Omega^1 A$ together with $d(I)$, hence the kernel of $\Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B$ it is equal to $d(I)$. ■

Exact sequence of Kähler differentials (2)

CLAIM 1: Let $\varphi: A \longrightarrow B$ be a surjective ring homomorphism, and I its kernel. Assume that φ admits a section $\sigma: B \longrightarrow A$. **Then the sequence**

$$0 \longrightarrow I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B \longrightarrow 0$$

is exact.

Proof: Consider the map $\rho: A \longrightarrow I/I^2$ taking x to $x - \sigma(\varphi(x))$. Since $(x - \sigma(\varphi(x)))(y - \sigma(\varphi(y))) \in I^2$, we have

$$xy + \sigma(\varphi(x))\sigma(\varphi(y)) = +x\sigma(\varphi(y)) + y\sigma(\varphi(x)).$$

Then

$$\begin{aligned} x(y - \sigma(\varphi(y))) + y(x - \sigma(\varphi(x))) &= 2xy - xy - \sigma(\varphi(x))\sigma(\varphi(y)) \\ &= xy - \sigma(\varphi(x))\sigma(\varphi(y)). \end{aligned}$$

Therefore, ρ is a derivation. Since ρ takes an element of I to itself, **the corresponding universal map $\Omega^1 A \otimes_A B \longrightarrow I/I^2$ is by construction inverse to δ.** This implies that $I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B$ is injective. ■

Zariski cotangent space

DEFINITION: Let $\mathfrak{m} \subset R$ be a maximal ideal. **The Zariski cotangent space** to $\operatorname{Spec} R$ in $\mathfrak{m}$ is $\frac{\mathfrak{m}}{\mathfrak{m}^2}$.

PROPOSITION: Let R be a ring over k, and $\mathfrak{m} \subset R$ be a maximal ideal which satisfies $\frac{R}{\mathfrak{m}} = k$. Then **the Zariski cotangent space $T^*_{\mathfrak{m}} \operatorname{Spec} R$ is equal to $\Omega^1_k R \otimes_R \frac{R}{\mathfrak{m}}$.**

Proof: Let $R_1 := \frac{R}{\mathfrak{m}}$. Since $\frac{R}{\mathfrak{m}} = k$, the map $R \longrightarrow R_1$ has a section. Then the exact sequence

$$0 \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \xrightarrow{\delta} \Omega^1_k R \otimes_R R^1 \xrightarrow{\varphi} 0 = \Omega^1_k R_1$$

implies that $\frac{\mathfrak{m}}{\mathfrak{m}^2} = \Omega^1 R \otimes_R R^1$. ∎

COROLLARY: If X is a smooth manifold, $x \in X$, and $\mathfrak{m}_x$ the ideal of all smooth functions vanishing in x. **Then $\mathfrak{m}_x^2$ is the ideal of all functions f such that $f(x) = 0$ and $df|_x = 0$.** In particular, **the natural map $\frac{\mathfrak{m}_x}{\mathfrak{m}_x{}^2} \longrightarrow T^*_x X$ mapping f to $df|_x$ is an isomorphism**.

This explains the term "Zariski cotangent space".

Regular points of an algebraic variety

PROPOSITION: Let $X \subset \mathbb{C}^n$ be an affine variety defined by a system of polynomial equations $f_1(z) = f_2(z) = ... = f_k(z) = 0$, and X_x a localization of $B = \mathcal{O}_B$ in the maximal ideal $\mathfrak{m}_x$ associated with a point $x \in X$. Denote by $W \subset T^*_x\mathbb{C}^n$ the space generated by the differentials $df_1, ..., df_k$. **Then $T^*_x X = \frac{T^*_x\mathbb{C}^n}{W}$.** Moreover, **$B_x$ is regular if and only if $\dim X = n - \dim W$.**

Proof: Let $A = \mathbb{C}[t_1, ..., t_n]$, and I the kernel of the natural surjection $A \longrightarrow B$. Consider the exact sequence $I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B \longrightarrow 0$. Then $\Omega^1 B \otimes_A \frac{A}{\mathfrak{m}_x} = \frac{\Omega^1 A \otimes_A \frac{A}{\mathfrak{m}_x}}{\delta(I/I^2)} = \frac{\langle dt_1, ..., dt_n\rangle}{\langle df_1, .., df_k\rangle}$. **Therefore, the Zariski cotangent space $T^*_x X = \Omega^1 B \otimes_A \frac{A}{\mathfrak{m}_x} = \frac{T^*_x\mathbb{C}^n}{W}$ is $(n - \dim_k W)$-dimensional.** On the other hand, **B_x is regular if and only if $\dim T^*_x X = \dim X$** (Theorem 3). ■

Regular points and regular values

COROLLARY: Let $X \subset \mathbb{C}^n$ be an affine variety defined by a system of polynomial equations $f_1(z) = f_2(z) = ... = f_k(z) = 0$. Suppose that 0 is a regular value of the map $F(z) = (f_1(z), f_2(z), ..., f_k(z))$. **Then $T_x^*X = \dim X$ for any $x \in X$.** In particular, **all localizations of X in maximal ideals are regular.**

Proof: Since X is given by k equations, dimension of X is at least $n - k$. On the other hand, $\dim T_x^*X = n - \dim W = n - k$ as shown above. By Theorem 3, $\dim T_x^*X \geqslant \dim X$, and **the equality is realized precisely when x is regular.** ■