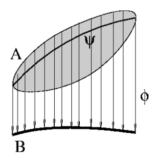
Handout 1: remedial set theory

Rules: This is a class assignment for the next week. Please solve as many exercises as you can, bring me what you can before the Wednesday week after. Wednesdays 17:00 we will discuss the solutions in a monitor session. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

1.1 The Axiom of Choice and its applications

Definition 1.1. Let $\phi : A \longrightarrow B$ be a surjective map of sets. **A section** of ϕ is a map $\psi : B \longrightarrow A$ such that $\psi \circ \phi = \mathsf{Id}$.



Definition 1.2. **Axiom of Choice** (AC) states that each surjective map admits a section.

Definition 1.3. **Partial order** is a relation $x \prec y$, which is **transitive** (if $x \prec y$ and $y \prec z$ then $x \prec z$) and **non-reflexive** ($z \prec z$ does not hold for any z). A set with partial order is called **partially ordered set**, or **poset**.

Definition 1.4. A partial order on S is called **total order** if for all $x \neq y$ either $x \prec y$ or $y \prec x$. A totally ordered poset S is called **well ordered** if any subset $S_1 \subset S$ has a minimal element (an element $v \in S_1$ such that $v \prec v'$ for all $v' \in S_1$ distinct from v).

Definition 1.5. Two posets A, B are called **isomorphic** if there exists a bijection from A to B preserving the order, with the inverse also preserving the order. Isomorphism classes of well ordered sets are called **ordinals**, or **ordinal numbers**. **Finite ordinals** are the same as natural numbers.

Remark 1.1. One can **add** the ordinals by taking a union $X \coprod Y$ of two well ordered sets and setting $X \prec Y$. Also, the ordinals can be multiplied: order on a product

$$(x, y) \prec (x', y') \text{ if } x \prec x', \text{ or } x = x', y \prec y'.$$

Exercise 1.1 (*). Prove that addition of ordinals is non-commutative.

Exercise 1.2. Prove that addition of ordinals is associative.

Exercise 1.3 (*). Prove that multiplication of ordinals is non-commutative.

Definition 1.6. **Interval** in a totally ordered set S is $[a, b] := \{x \in S \mid a \prec x \prec b\}$ **Initial interval** is an interval starting from the minimal element.

Exercise 1.4 (!). Let X, Y be well ordered sets. Prove that X is isomorphic a to an initial interval of Y, or Y is isomorphic a to an initial interval of X.

Exercise 1.5. Prove that such an isomorphism is unique.

Definition 1.7. Zermelo theorem states that any set admits well ordering.

Exercise 1.6. Deduce the Axiom of Choice from the Zermelo theorem.

Hint. The section ψ will map $b \in B$ to the minimal element in $\phi^{-1}(b)$.

Definition 1.8. Let $(S, \prec)$ be a poset. An element $x \in S$ is called **maximal** if there is no $y \in S$ such that $x \prec y$. For a subset $S_1 \subset S$ and $x \in S$, we write $S_1 \prec x$ if $\xi \prec x$ for all $\xi \in S_1$. **Zorn lemma** states that any poset $(S, \prec)$ has a maximal element if for any well ordered subset $S_1 \subset S$ there exists $x \in S$ such that $S_1 \backslash \{x\} \prec x$.

Exercise 1.7 (!). Deduce Zermelo theorem from Zorn lemma.

Hint. Let A be a set which we want to endow with a well ordering. Take for S the set of subsets of A equipped with well ordering, and write $S_1 \prec S_2$ if S_1 is an initial interval of S_2. Prove that the poset S satisfies assumptions of Zorn lemma, and any maximal element of S is A with well ordering.

Exercise 1.8 (!). Let A be a set, and S the set of all subsets $A_0 \subset A$ equipped with a well ordering $\prec_{A_0}$. Assume that S does not contain A. We intend to deduce Zermelo theorem from Axiom of Choice by absurd.

a. Using Axiom of Choice, construct a map $\phi: S \longrightarrow A$ mapping $W \in S$ to an element x in the complement $A \backslash W$.

b. Let $R \subset S$ be the set of all well ordered subsets $(W, \prec_W) \subset A$, starting from $\emptyset$, such that for any initial interval $W_0 \subset W$, the minimal element of the complement $W \backslash W_0$ is $\phi(W_0)$. Prove that R is well ordered by inclusion, where $X \prec Y$ if and only if $X \in R$ is an initial interval of $Y \in R$.

c. Prove that the union of all $W \in R$ is well ordered.

Exercise 1.9 (!). Deduce Zorn lemma from Axiom of Choice, as follows. Like for Zermelo theorem, we prove Zorn lemma ad absurdum. Let $(A, \prec)$ be a poset without a maximal element, and S the set of well ordered subsets of A.

a. Prove that for any well ordered $W \subset A$, there exists $x \in A$ which satisfies $W \prec x$. Using Axiom of Choice, find a function $\phi: S \longrightarrow A$ such that $W \prec \phi(W)$ for all W, starting from $\emptyset \in S$.

b. Let $R_\phi \subset S$ be the set of all well ordered subsets $W \subset A$ such that for any initial interval $W_0 \subset W$, the minimal element of $W \backslash W_0$ is $\phi(W_0)$. Prove that R is well ordered by inclusion.

c. Prove that the union of all $W \in R$ is well ordered.

1.2 Hamel basis

Exercise 1.10. Let A, B be two sets. Prove that A is equinumerous to a subset of B, or B is equinumerous to a subset of A.

Hint. Use Zermelo theorem.

Exercise 1.11 (!). (Cantor-Bernstein-Schroeder theorem)
Suppose that A is equinumerous to a subset of B, and B is equinumerous to a subset of A. Prove that A is equinumerous to B.

Remark 1.2. Cantor-Bernstein-Schroeder theorem is in fact independent from the Axiom of Choice. Please google its proof, if you never saw it, it is not very complicated, but hard to invent (Cantor spent 20 years trying to prove it, did not succeed, and became very depressed in result).

Exercise 1.12 (!). Let X be an infinite set. Prove that $X \times \mathbb{Z}$ is equinumerous to X.

Hint. Use Zermelo theorem.

Exercise 1.13. As elsewhere, all rings are assumed to have unity. Let $I \subset R$ be an ideal in a ring. Prove that I is contained in a maximal ideal.

Definition 1.9. Let V be a vector space. **Hamel basis**, or just **basis** in V is a maximal set of linearly independent vectors in V.

Exercise 1.14. Prove that any vector space has a basis.

Exercise 1.15. Prove that any vector space over a countable field is equinumerous with its basis.

Exercise 1.16. (!) Let S_1, S_2 be bases in an infinite-dimensional vector space W over a field k, and $A \in k^{S_1 \times S_2}$ the transition matrix expressing S_1 through S_2. Denote by $T \subset S_1 \times S_2$ the set of pairs of indices $\alpha \in S_1, \beta \in S_2$ such that the corresponding coefficient $a_{\alpha,\beta}$ in A is non-zero.

a. Prove that the projection functions $\pi_i : T \longrightarrow S_i$ are surjective and preimage $\pi_1^{-1}(x)$ is finite for any $x \in S_i$.

b. Use this to show that T is equinumerous to S_1.

c. Construct a surjective map $T \to S_2$.

d. Prove that S_1 is equinumerous to S_2.

Hint. Construct injective maps $S_1 \hookrightarrow T \hookrightarrow S_1^{\mathbb{Z}}$, prove that $S_1^{\mathbb{Z}}$ is equinumerous to S_1, and apply Cantor-Bernstein-Schroeder.