

Home assignment 3: Tensor product

Rules: This is a class assignment for the next week. Please solve as many exercises as you can, bring me what you can before the Wednesday week after. Wednesdays 17:00 we will discuss the solutions in a monitor session. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

3.1 Tensor product of R -modules

Remark 3.1. All rings are assumed to be commutative and with unity.

Definition 3.1. Let R be a ring, and M, M' modules over R . We denote by $M \otimes_R M'$ an R -module generated by symbols $m \otimes m'$, $m \in M, m' \in M'$, modulo relations $r(m \otimes m') = (rm) \otimes m' = m \otimes (rm')$, $(m + m_1) \otimes m' = m \otimes m' + m_1 \otimes m'$, $m \otimes (m' + m'_1) = m \otimes m' + m \otimes m'_1$ for all $r \in R, m, m_1 \in M, m', m'_1 \in M'$. Such an R -module is called **the tensor product of M and M' over R** .

Definition 3.2. Let M_1, M_2, M be modules over a ring R . **Bilinear map** $\mu(M_1, M_2) \xrightarrow{\phi} M$ is a map satisfying $\phi(rm, m') = \phi(m, rm') = r\phi(m, m')$, $\phi(m + m_1, m') = \phi(m, m') + \phi(m_1, m')$, $\phi(m, m' + m'_1) = \phi(m, m') + \phi(m, m'_1)$.

Exercise 3.1. (universal property of tensor product)

Construct a natural bijective correspondence between the set of homomorphisms $\text{Hom}_R(M_1 \otimes_R M_2, M)$ and the set of bilinear maps $\text{Bil}(M_1 \times M_2, M)$.

Exercise 3.2. Find non-zero R -modules A, B , such that $A \otimes_R B = 0$ for

- $R = \mathbb{C}[t]$.
- R is the ring of complex analytic functions on \mathbb{C} .
- $R = \mathbb{Z}$.
- $R = \mathbb{Z}/10$.

Definition 3.3. Let M, M' be R -modules. Consider the group $\text{Hom}_R(M, M')$ of R -module homomorphisms. We consider $\text{Hom}_R(M, M')$ as an R -module, using $r\phi(m) := \phi(rm)$. This R -module is called **internal Hom functor**, denoted $\mathcal{H}om_R$.

Exercise 3.3. Prove that $\mathcal{H}om_R(M_1 \otimes_R M_2, M) = \mathcal{H}om_R(M_1, \mathcal{H}om_R(M_2, M))$.

Exercise 3.4. Let $B : M_1 \times M_2 \rightarrow M$ be a bilinear map of R -modules.

- Prove that there exists a unique homomorphism $b : M_1 \otimes M_2 \rightarrow M$, making the following diagram commutative:

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{B} & M_1 \otimes M_2 \\ & \searrow \tau & \downarrow b \\ & & M \end{array}$$

- Prove that this property defines $M_1 \otimes M_2$ uniquely.

3.2 Exact sequences and $\mathcal{H}om$ -functor

Definition 3.4. **Exact sequence** of R -modules is a sequence of homomorphisms

$$\dots \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+2} \longrightarrow \dots$$

(finite or infinite) such that the kernel of n -th arrow is the image of $n - 1$ -th arrow for all n . **Short exact sequence** is an exact sequence of form $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$.

Exercise 3.5. Let $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R -modules. Prove that the sequence

$$0 \longrightarrow \mathcal{H}om_R(M_3, N) \longrightarrow \mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N)$$

is exact, for any R -module N .

Exercise 3.6. Let $R = \mathbb{Z}$. Find an example of an injective homomorphism $M_1 \longrightarrow M_2$ such that $\mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N)$ is not surjective for some R -module N .

Definition 3.5. An **exact functor** F is a functor on the category of R -modules mapping any exact sequence of R -modules to an exact sequence.

Exercise 3.7 (*). Let F be a functor which maps any short exact sequence to an exact sequence. Prove that F is exact.

Definition 3.6. An R -module is called **injective** if the functor

$$M \longrightarrow \mathcal{H}om_R(M, N)$$

is exact.

Exercise 3.8. Find a non-zero injective module over \mathbb{Z} .

Exercise 3.9 (*). Find a non-zero injective module over \mathbb{Z} containing $\mathbb{Z}/n\mathbb{Z}$.

Exercise 3.10. Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$ be an exact sequence of R -modules. Prove that the corresponding sequence

$$0 \longrightarrow \mathcal{H}om_R(N, M_1) \longrightarrow \mathcal{H}om_R(N, M_2) \longrightarrow \mathcal{H}om_R(N, M_3)$$

is exact for any R -module N .

Exercise 3.11. Let $R = \mathbb{Z}$. Find an epimorphism (surjective homomorphism) of R -modules $M_2 \longrightarrow M_3$ such that the corresponding map

$$\mathcal{H}om_R(N, M_2) \longrightarrow \mathcal{H}om_R(N, M_3)$$

is not surjective for some R -module N .

Definition 3.7. An R -module M is called **projective** if the functor $M \rightarrow \mathcal{H}om_R(N, M)$ is exact.

Exercise 3.12 (!). Prove that a finitely generated R -module N is projective if and only if it is a direct summand of a free module R^n , that is, there is a direct sum decomposition of R -modules $R^n = N \oplus N'$.

Hint. Consider the surjective homomorphism

$$\mathcal{H}om_R(N, M_2) \rightarrow \mathcal{H}om_R(N, M_3)$$

where $M_3 = N$, and M_2 is a free module equipped with a surjective homomorphism to N .

Exercise 3.13 (*). Suppose that R is a finitely generated ring over \mathbb{C} , and any finitely generated R -module is projective. Prove that R is a direct sum of fields, or find a counterexample.

3.3 Tensor product and exact sequences

Exercise 3.14. Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of R -modules. Prove that for any R -modules N, N' , the sequence

$$\begin{aligned} 0 \rightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_3, N)) \rightarrow \\ \rightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_2, N)) \rightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_1, N)) \end{aligned}$$

is exact

Exercise 3.15. Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of R -modules. Prove that for any R -modules N, N' , the sequence

$$0 \rightarrow \mathcal{H}om_R(M_3 \otimes N', N) \rightarrow \mathcal{H}om_R(M_2 \otimes N', N) \rightarrow \mathcal{H}om_R(M_1 \otimes N', N)$$

is exact.

Definition 3.8. A **complex** of R -modules is a sequence $M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \xrightarrow{d_3} \dots$ such that $d_i \circ d_{i+1} = 0$. **Cohomology** of this complex are the quotient groups, $H^i(M_*) := \frac{\ker(d_i)}{\text{im } d_{i-1}}$. A complex is **exact** if it has zero cohomology.

Exercise 3.16. Consider a complex $M_1 \xrightarrow{\mu} M_2 \xrightarrow{\rho} M_3 \rightarrow 0$ of R -modules such that the corresponding sequence

$$0 \rightarrow \mathcal{H}om_R(M_3, N) \xrightarrow{\rho_N} \mathcal{H}om_R(M_2, N) \xrightarrow{\mu_N} \mathcal{H}om_R(M_1, N) \quad (3.1)$$

is exact for any R -module N . Prove that E is also exact

Hint. Use injectivity of ρ_N to prove surjectivity of ρ by setting $N := M_3 / \text{im } \rho$. To prove exactness of E in the second term, use $N = M_2 / \text{im } \mu$ and apply exactness of the sequence (3.1) in the second term.

Exercise 3.17 (*). Let $E = \left(\dots \rightarrow M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \rightarrow \dots \right)$ be a complex of R -modules such that

$$\dots \rightarrow \mathcal{H}om_R(M_3, N) \rightarrow \mathcal{H}om_R(M_2, N) \rightarrow \mathcal{H}om_R(M_1, N) \rightarrow \dots$$

is exact for all R -modules N . Prove that E is also exact.

Exercise 3.18 (!). Let $M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of R -modules. Prove that the sequence

$$M_1 \otimes_R M \rightarrow M_2 \otimes_R M \rightarrow M_3 \otimes_R M \rightarrow 0$$

is also exact.

Hint. Use Exercise 3.16, and apply exactness of

$$0 \rightarrow \mathcal{H}om_R(M_3 \otimes M, N) \rightarrow \mathcal{H}om_R(M_2 \otimes M, N) \rightarrow \mathcal{H}om_R(M_1 \otimes M, N),$$

using an isomorphism $\mathcal{H}om_R(M_1 \otimes_R M_2, M) = \mathcal{H}om_R(M_1, \mathcal{H}om_R(M_2, M))$ (Exercise 3.3).

Exercise 3.19 (!). Let $R = \mathbb{C}[t]$. Find an exact sequence of R -modules $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ and an R -module N such that the corresponding map $M_1 \otimes_R M \rightarrow M_2 \otimes_R M$ is not injective.

Exercise 3.20 (!). Let $I \subset R$ be an ideal. Prove that for any R -module M , one has $M \otimes_R (R/I) \cong M/IM$.

Hint. Take an exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ and apply the functor $\otimes M$.

Exercise 3.21 (!). Let I, I' be distinct maximal ideals in a ring R . Prove that $R/I \otimes_R R/I' = 0$.

Exercise 3.22 (!). Let $R^n \rightarrow R^m$ be a surjective homomorphism of R -modules. Prove that $n \geq m$.