# Home assignment 7: Representations of finite groups

**Rules:** This is a class assignment for the next week. Please solve as many exercises as you can, bring me what you can before the Wednesday week after. Wednesdays 17:00 we will discuss the solutions in a monitor session. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

**Definition 7.1. Representation of a group** $G$, or $G$**-representation** is a group homomorphism $G \longrightarrow GL(V)$. **Irreducible representation** is a representation having no proper $G$-invariant subspaces. **Semisimple representation** is a direct sum of irreducible ones.

**Exercise 7.1.** Consider the group $G$ of non-degenerate upper triangular matrices $n \times n$. Construct a non-semisimple representation of $G$.

**Exercise 7.2.** Construct a finite group $G$ and a non-semisimple representation $G \longrightarrow GL(V)$, where $V$ is a finite-dimensional vector space over a finite field.

**Definition 7.2.** Let $V$ be a finite-dimensional irreducible group representation over $\mathbb{R}$. It is called **real** if $\operatorname{End}_G(V) = \mathbb{R}$, **complex** if $\operatorname{End}_G(V) = \mathbb{C}$, and **quaternionic** if $\operatorname{End}_G(V)$ is the algebra of quaternions.

**Exercise 7.3.** Find an example of a quaternionic representation of a finite group.

**Exercise 7.4.** Find a quaternionic representation of a cyclic group of order 17, or prove that such a representation cannot exist.

**Exercise 7.5 (4 points).** Find a quaternionic representation of the group $G$ of isometries of a regular tetrahedron, or prove that such a representation cannot exist.

**Exercise 7.6.** Let $V = \mathbb{Q}^2$ be an irreducible 2-dimensional representation of the cyclic group $G = \mathbb{Z}/n$ over $\mathbb{Q}$.

a. Prove that the algebra $\operatorname{End}_G(V)$ of endomorphisms of this representation is commutative and 2-dimensional.

b. (2 points) Prove that $\operatorname{End}_G(V)$ is a number field isomorphic to $\mathbb{Q}[\sqrt{a}]$ for some $a \in \mathbb{Z}$.