Home assignment 10: Poincaré series

Rules: This is a class assignment for the next week. Please solve as many exercises as you can, bring me what you can before the Wednesday week after. Wednesdays 17:00 we will discuss the solutions in a monitor session. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

Exercise 10.1. Let $n \mapsto H(n)$ be a polynomial of degree d. Prove that $n \mapsto H(n) - H(n-1)$ is a polynomial of degree $d-1$.

Exercise 10.2. Let $p(t) = \sum a_i t^i$, and $q(t) = \sum t^{ik} \cdot \sum a_i t^i = \sum b_i(t)$, for some fixed integer $k > 0$. Assume that $a_i = h(i)$, where $h(i)$ for $i \gg 0$ is a polynomial of degree d. Prove that for any $s \in \mathbb{Z}$, $h_1(i) := b_{ik+s}$ is a polynomial of degree $d+1$ when $i \gg 0$.

Hint. Use the previous exercise.

Exercise 10.3. Let $f(t)$ be a polynomial, $k_1, ..., k_n \in \mathbb{Z}^{>0}$, and $p(t)$ a rational function $p(t) = f(t) \prod_{i=1}^n (1 - t^{k_i})^{-1}$. Consider the Taylor series $p(t) = \sum a_i t^i$. Let $k := \prod_{i=1}^n k_i$ and $s \in \mathbb{Z}$. Prove that there is a polynomial $H(i)$ such that $a_{ki+s} = H(i)$ for $i \gg 0$.

Hint. Use the previous exercise

Definition 10.1. Let $\mathcal{C}$ be the category of finite generated graded R-modules, where R is a graded ring. A function $P : \mathcal{C} \to \mathbb{Z}[[t]]$ is called **additive** if for any exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, we have $P(M_1) + P(M_2) = P(M_3)$.

Exercise 10.4. Let P be an additive function on the category $\mathcal{C}$ of finite generated graded modules over polynomials $R = \mathbb{C}[z_1, ..., z_n]$. Assume that $P(M)$ is polynomial for any R-module of finite dimension over $\mathbb{C}$, and $P(M_d) = t^d P(M)$, where M_d is obtained from M by shifting the grading by $+d$. Prove that $P(M)$ is a rational function of form $g(t) + f(t) \prod_{i=1}^n (1 - t^{k_i})^{-1}$ for some polynomials $g(t)$, $f(t)$.

Hint. Use the previous exercise.

Definition 10.2. The Poincaré series of a graded module M^* over polynomials is the function $P(t) := \sum_{i=0}^\infty h_i t^i$, where $h_i := \dim M^i$.

Exercise 10.5. Prove that the Poincaré series of $C[x, y]$ is $\frac{1}{(1-t)^2}$.

Exercise 10.6. Prove that the Poincaré series of $C[x, y]/(x^2)$ is $\frac{(t+1)}{(1-t)}$.

Exercise 10.7. Prove that the Poincaré series of the ideal $(x^2, y^3) \subset C[x, y]$ is $\frac{(t^2+t^3-t^5)}{(1-t)^2}$.

Exercise 10.8 (*). Let $P(t) := \sum_{i=0}^\infty h_i t^i$. Assume that for some $k \in \mathbb{Z}^{>0}$ the map $i \mapsto h_{ki+s}$ is a polynomial function for any $s \in \mathbb{Z}$ and $i \gg 0$. Prove that $P(t)$ is rational, or find a counterexample.