Commutative algebra, final exam (2026)

Rules: Every student receives from me a list of 10 exercises (chosen randomly), and has to solve as many of them as you can before September 2024. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solutions, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes).

The final score N is obtained by summing up the points from the exam problems and the class tests, using the formula $N = e + t/10$, where t is the sum of class tests, e the points for exam problems. Marks: C when $30 \leqslant N < 49$, B when $50 \leqslant N < 69$, A when $70 \leqslant N \leqslant 89$, A+ when $N > 90$.

1 Hilbert's Nullstellensatz and Zariski topology

Definition 1.1. Boolean ring is a ring such that all its elements are idempotents, that is, satisfy $a^2 = a$.

Exercise 1.1 (10 points). Prove that all prime ideals in a boolean ring are maximal, or find a counterexample.

Definition 1.2. Spectrum of a ring A is the set of all prime ideals of A. **Zariski topology** on the spectrum is defined by the base $\{U_f\}$ of open sets, enumerated by $f \in A$, defined as follows: a prime ideal $\mathfrak{p}$ belongs to U_f if $f \notin \mathfrak{p}$. **Maximal spectrum** is the set of maximal ideals with the same topology.

Exercise 1.2 (10 points). Prove that the spectrum of a boolean ring with Zariski topology is Hausdorff.

Exercise 1.3 (30 points). Let M be a compact, connected manifold of positive dimension, and $A = C(M)$ the ring of continuous functions on M. Prove that any prime ideal of A is maximal, or find a counterxample.

Exercise 1.4 (20 points). Let $P(x, y)$ be an irreducible polynomial, and $R := \frac{\mathbb{C}[x,y]}{(P)}$. Prove that all ideals in R are principal, or find a counterexample.

Exercise 1.5 (10 points). Let $A = \mathbb{R}[t]$, and X be its maximal spectrum. Prove that X with its Zariski topology is homeomorphic to the maximal spectrum of $\mathbb{C}[t]$.

Exercise 1.6 (30 points). Let $R := \frac{\mathbb{R}[x,y]}{(x^2+y^2-1)}$. Prove that all ideals in R are principal, or find a counterexample.

2 Noetherian rings

Remark 2.1. In this section, the rings are not finitely-generated or Noetherian unless noted otherwise.

Definition 2.1. A module is **finitely representable** if it is a quotient of a free module by a finitely generated module.

Exercise 2.1 (10 points). Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of A-modules, and M_1, M_3 are finitely representable. Prove that M_2 is also finitely representable.

Definition 2.2. A prime ideal is called **minimal** if it does not contain any smaller prime ideals.

Exercise 2.2 (10 points). Let A be a finitely generated ring over $\mathbb{C}$. Prove that the number of minimal prime ideals in A is finite.

Exercise 2.3 (10 points). Let M be an A-module, where A is not necessarily Noetherian, and $N_1, N_2 \subset M$ submodules which satisfy $N_1 \cap N_2 = 0$. Assume that the modules M/N_1 and M/N_2 are Noetherian. Prove that M is Noetherian.

Exercise 2.4 (10 points). Let A be a ring without zero divisors. Suppose that any descending chain of ideals in A stabilizes. Prove that A is a field.

Exercise 2.5 (10 points). Let $M = \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[t], \mathbb{C})$, equipped with a natural structure of $\mathbb{C}[t]$-module. Is M finitely generated over $\mathbb{C}[t]$?

3 Irreducible polynomials and Galois theory

Exercise 3.1 (20 points). An R-module is called **projective** if it is a direct summand of a free R-module. Let R be a Noetherian ring. Suppose that all finitely generated R-modules are projective. Prove that R is a direct sum of fields, or find a counterexample.

Exercise 3.2 (10 points). Let $k = \mathbb{C}(t)$ be the field of rational functions on $\mathbb{C}$. Prove that for any n there exists an irreducible polynomial $P(t) \in K[t]$ of degree n.

Exercise 3.3 (10 points). Prove that the polynomial $x^3 + y + y^5$ is irreducible in $\mathbb{Z}[x, y]$.

Exercise 3.4 (10 points). Let $K := \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}]$. Prove that $[K : \mathbb{Q}]$ is a Galois extension, and find its Galois group.

Exercise 3.5 (10 points). Prove that for any $d \in \mathbb{Z}^{>0}$ there exists an irreducble polynomial $P \in \mathbb{C}[x, y]$, such that $P(x, y) = 0$ defines a smooth submanifold in $\mathbb{C}^2$.

Exercise 3.6 (10 points). Let A be a finitely generated ring over $\mathbb{R}$, and $I \subset A$ a maximal ideal. Prove that A/I is isomorphic to $\mathbb{C}$ or $\mathbb{R}$.

4 Tensor product of R-modules

Definition 4.1. A module M over a ring R is called **flat** if the functor $X \longrightarrow X \otimes_R M$ is exact.

Definition 4.2. Let A be a ring without zero divisors. **Torsion** in an A-module M is the kernel of a natural map $M \longrightarrow M \otimes_A k(A)$.

Exercise 4.1 (10 points). Let M be a flat R-module, where R is a ring without zero divisors. Prove that M is torsion-free.

Exercise 4.2 (20 points). Let $M_1 \subset M_2 \subset ...$ be a sequence of embedded R-modules. Assume that all M_i are flat. Prove that $\bigcup M_i$ is also flat.

Exercise 4.3 (20 points). Prove that a finitely generated module over a Noetherian local ring is flat if and only if it is free.

Exercise 4.4 (20 points). Let A be a ring without zero divisors, and M an A-module which is flat over all local rings $A_s \supset A$ obtained by localizing A in prime ideals. Prove that M is flat.

Exercise 4.5 (30 points). Prove that any finitely generated module over a Boolean ring is flat.

5 Tensor product of rings

Exercise 5.1 (10 points). Let R_1, R_2 be rings over $\mathbb{C}$ such that $R_1 \otimes_{\mathbb{C}} R_2$ is finitely generated. Prove that R_1, R_2 are finitely generated, or find a counterexample.

Definition 5.1. An R-module M is called **invertible** if $M \otimes_R M^* \cong R$, where $M^* = \operatorname{Hom}_R(M, R)$.

Exercise 5.2 (10 points). Prove that each finitely generated, invertible R-module admits a monomorphism to a free R-module.

Exercise 5.3 (10 points). Let P be the set of isomorphism classes of invertible R-modules. Prove that tensor product defines a group structure on P (this group is called **Picard group** of P). Find P for the ring $\mathbb{C}[t]$.

Exercise 5.4 (10 points). Let M, N be finitely generated modules over the ring $\mathbb{Z}_p$ of p-adic integers, such that $M \otimes_{\mathbb{Z}_p} N = 0$. Prove that $M = 0$ or $N = 0$.

Exercise 5.5 (10 points). Let A be the ring of complex-analytic functions on $\mathbb{C}$, and $k(A)$ its fraction field. Prove that the transcendence degree of A over $\mathbb{C}$ is infinite.

6 Normality, irreducibility, smoothness

Exercise 6.1 (10 points). Let A be an integrally closed ring, G a finite group acting on A by automorphisms, and A^G the ring of invariants. Prove that A^G is integrally closed.

Exercise 6.2 (10 points). Prove that the ring $\mathbb{C}[x,y,z]/(y^2 - xz)$ is integrally closed.

Exercise 6.3 (20 points). Let $\zeta_n \in \mathbb{C}$ be a primitive root of unity of degree n. Define the action of the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ with coordinates x, y in such a way that the generator t maps x to $\zeta_n x$ and y to $\zeta_n^{-1} y$. Prove that the quotient $\mathbb{C}^2/G$ is singular.

Exercise 6.4 (20 points). Let $A \subset \mathbb{C}^n$ be a subvariety given by equations $z_1^{p_1} = z_2^{p_2} = ... z_n^{p_n}$. Suppose that all p_i are different primes. Prove that A is irreducible.

Exercise 6.5 (10 points). Let $P(x,y,z)$ and $Q(x,y,z) \in \mathbb{C}[x,y,z]$ be irreducible, coprime polynomials, and X a variety defined by $P = Q = 0$. Prove that X is irreducible, or find a counterexample.

Exercise 6.6 (20 points). (removed)

Exercise 6.7 (20 points). (removed)

Exercise 6.8 (10 points). Let $S \subset \mathbb{C}^2$ be a smooth hypersurface defined by an irreducible quadratic equation. Prove that S is isomorphic to $\mathbb{C}$ or to $\mathbb{C} \backslash \{0\}$.

Exercise 6.9 (20 points). Let $S \subset \mathbb{C}^n$ be a smooth hypersurface given by a quadratic equation. Prove that S is normal.